* **Def:** A <u>k-form</u> on an open subset $U \subset \mathbb{R}^n$ is a function with values in $\Lambda^k T^*$:
$$\omega = \sum_{i_1<\dots<i_k} p_{i_1\dots i_k}(x)\, dx_{i_1}\wedge\dots\wedge dx_{i_k}. \quad \left(\text{also denoted } = \sum_{|I|=k} p_I\, dx_I\right)$$

The space of C^∞ k-forms on $U\subset\mathbb{R}^n$: $\Omega^k(U) = C^\infty(U, \Lambda^k T^*)$.

* <u>Exterior product</u> $(f\, dx_{i_1}\wedge\dots\wedge dx_{i_k})\wedge(g\, dx_{j_1}\wedge\dots\wedge dx_{j_l}) = (fg)\, dx_{i_1}\wedge\dots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\dots\wedge dx_{j_l}$

$dx_i\wedge dx_j = -dx_j\wedge dx_i$ $\qquad (=0$ if $I\cap J\neq\emptyset$, $=\pm(fg)\, dx_{I\sqcup J}$ if $I\cap J=\emptyset)$.

* The <u>exterior derivative</u> $d: \Omega^k \to \Omega^{k+1}$ is $d\left(\sum_I p_I\, dx_I\right) = \sum_{I,j} \frac{\partial p_I}{\partial x_j}\, dx_j\wedge dx_I$.

<u>Eg</u>: $\Omega^0\to\Omega^1$: $df = \sum \frac{\partial f}{\partial x_i}\, dx_i$.

$\Omega^1(\mathbb{R}^2)\to\Omega^2(\mathbb{R}^2)$, $d(p\, dx + q\, dy) = \left(-\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right) dx\wedge dy$.

<u>Prop</u>: $d^2 = 0$ ie. $\forall\omega\in\Omega^k$, $d(d\omega)=0$.

* <u>Pullback of differential forms</u>: if $\varphi: U\to V$ is a smooth map $(U\subset\mathbb{R}^n, V\subset\mathbb{R}^m)$ then we have a map $\varphi^*: \Omega^k(V)\to\Omega^k(U)$ characterized by

$$\begin{cases} (1) & \text{for functions } (k=0),\ \varphi^*(f) = f\circ\varphi \\ (2) & \varphi^*(\alpha\wedge\beta) = \varphi^*\alpha\wedge\varphi^*\beta \\ (3) & \varphi^*(d\alpha) = d(\varphi^*\alpha). \end{cases}$$

Concretely, denoting by (x_i) coords. on U, (y_j) on V, $\varphi^*(dy_j) = d(y_j\circ\varphi) = \sum_i \frac{\partial\varphi_j}{\partial x_i}\, dx_i$ $(= d\varphi_j)$

and $\varphi^*\left(\sum_J p_J(y)\, dy_{j_1}\wedge\dots\wedge dy_{j_k}\right) = \sum_J p_J(\varphi(x))\, \underbrace{d\varphi_{j_1}\wedge\dots\wedge d\varphi_{j_k}}_{=\sum_I \det\left(\frac{\partial(\varphi_{j_1},\dots,\varphi_{j_k})}{\partial(x_{i_1},\dots,x_{i_k})}\right) dx_I}$

Especially: for $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ and $k=n$,
$$\varphi^*(dy_1\wedge\dots\wedge dy_n) = (\det D\varphi)\, dx_1\wedge\dots\wedge dx_n$$

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* <u>Integration of differential forms</u>:

given $\omega = \sum_I p_I(x)\, dx_I \in \Omega^k(U)$, we can integrate ω over a k-dimensional submanifold $M\subset U$ parametrized by a smooth map from a k-cell $D\subset\mathbb{R}^k$ to $U\subset\mathbb{R}^n$ (or any other nice enough domain for integration), $\varphi: D\hookrightarrow U$, $M=\varphi(D)$,
$t\mapsto(\varphi_1(t),\dots,\varphi_n(t))$

by setting
$$\boxed{\int_M \omega = \int_D \sum_I p_I(\varphi(t))\, \det\left(\left(\frac{\partial\varphi_i}{\partial t_j}\right)_{\substack{i\in I\\ 1\le j\le k}}\right) |dt|.}$$

<u>check</u>: for 1-forms this agrees with path integral formula $\int_\gamma p_i\, dx_i = \int p_i(\gamma(t))\frac{dx_i}{dt}\, dt$

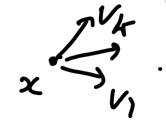
What this formula means is:

$$\begin{cases} \bullet \text{ for } n\text{-forms on } D \subset U \subset \mathbb{R}^n, & \int_D f\, dx_1 \wedge \dots \wedge dx_n = \int_D f\,|dx|. \\ \bullet \text{ for general } \varphi : D^k \to U \subset \mathbb{R}^n, & \int_{\varphi(D)} \omega = \int_D \underbrace{\varphi^* \omega}_{} \end{cases}$$

$\leftarrow$ k-form on $D \subset \mathbb{R}^k$ $\Rightarrow$ usual integral.

* Can similarly integrate k-forms over M = finite union of parametrized pieces.

* <u>Conceptually</u>: a k-form is a function with values in $\Lambda^k T^*$ = alternating multilinear forms on tangent vectors, ie. can evaluate $\omega(x)(v_1, \dots, v_k)$



This gives (for $|v_i| \to 0$) an approximation of the integral of ω over the small parallelepiped $P = \{x + \sum t_i v_i \,/\, (t_i) \in [0,1]^k\}$, as can be seen parametrizing P by $(t_i) \mapsto (x + \sum t_i v_i)$ and pullback. The definition of $\int_M \omega$ via pullback + Riemann integral on D amounts to subdividing M into approximate parallelepipeds $\varphi(Q_i)$, Q_i cubes $\subset D$, evaluating ω on each, and summing.

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* **General <u>pullback formula</u>**: given a smooth map $\varphi : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$, $\omega \in \Omega^k(V)$, and $M^k \subset U$: $\underline{\int_{\varphi(M)} \omega = \int_M \varphi^* \omega}$.

This is basically equivalent to change of variables formula for usual $\int_D f\,|dx|$, and implies that $\int_M \omega$ is independent of the manner in which we parametrize M as the image of a map $\varphi : D \to U$ (or union of pieces) as long as all reparametrizations are <u>orientation-preserving</u>
(ie. we compose $\varphi : D \to U$ with a diffeomorphism $g : \underset{\mathbb{R}^k}{\overset{\cap}{D'}} \to \underset{\mathbb{R}^k}{\overset{\cap}{D}}$ st. $\det(Dg) > 0$ everywhere).

<u>Ex</u>: $\omega = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$ on $\mathbb{R}^2 - \{0\}$, C_r = circle of radius r, oriented counterclockwise: (as path $(r,0) \to (r,0)$)

Pulling back via $\varphi : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, (polar coordinates),

$$\varphi^* \omega = \frac{(r\cos\theta)(\sin\theta\, dr + r\cos\theta\, d\theta) - (r\sin\theta)(\cos\theta\, dr - r\sin\theta\, d\theta)}{r^2} = d\theta$$

$$\text{So } \int_{C_r} \omega = \int_{\{r\} \times [0,2\pi]} \varphi^* \omega = \int_0^{2\pi} d\theta = 2\pi \qquad \text{(independent of } r\text{)}$$

(more directly, could just pullback via $\varphi : t \mapsto (\cos t, \sin t)$, $\varphi^* \omega = dt$ ...)

<u>Note</u>: $d\omega = 0$ (by direct calc., or using $\varphi^*(d\omega) = d(\varphi^* \omega) = d(d\theta) = 0 \underset{\varphi \text{ diffeo}}{\Rightarrow} d\omega = 0$)

i.e. ω is closed; but not exact! if $\exists f(x,y)$ on $\mathbb{R}^2-\{0\}$ st $df=\omega$ then path integral $\int_{C_r}\omega = \int_{C_r} df = f(r,0)-f(r,0)=0$. $\quad H^1_{dR}(\mathbb{R}^2-0)\neq 0$.

But... path integral is independent of radius r, or in fact same for any loop around the origin.

This is a consequence of <u>Stokes' theorem</u>.

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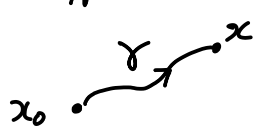
for $M\subset\mathbb{R}^n$ parametrized as $\varphi(D)$, $D\subset\mathbb{R}^k$ k-cell (or other nice domain) define $\partial M = (k-1)$-dimensional boundary $\varphi(\partial D)$ (for $D=\prod[a_i,b_i]$ a k-cell, this consists of $2k$ pieces...), with suitable orientation.

(most relevant to us: $\partial(\text{square}) = \text{its boundary, oriented counterclockwise}$).

<u>Stokes' thm:</u> $\quad \forall\omega\in\Omega^{k-1}$, $\displaystyle\int_M d\omega = \int_{\partial M}\omega$.

Application: if ω is a closed 1-form on a simply connected $U\subset\mathbb{R}^n$, the path-integral



$\int_\gamma \omega$ is indept of choice of path γ from base point x_0 to x.

Indeed, path-independence comes from Stokes for the surface S traced by a path homotopy:

$$d\omega=0 \Rightarrow \quad 0=\int_S d\omega = \int_{\partial S=\gamma-\gamma'}\omega = \int_\gamma\omega - \int_{\gamma'}\omega$$

So we can define $f(x)=\int_\gamma\omega$ for any path $\gamma: x_0\to x$.

Stokes again (= fund. thm. calc.) gives $\int_\gamma df = f(x)-f(x_0) = \int_\gamma \omega \quad \forall$ path γ,

and we find that $\omega = df$ is exact. ($\Rightarrow$ Poincaré lemma).

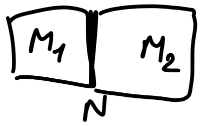
<u>Rmk:</u> Stokes' theorem for diff. forms in $\mathbb{R}^2$ and $\mathbb{R}^3$ specializes to all the theorems of multivariable calculus

$$\left(\begin{array}{l} k=0: \text{fund. thm. of calc. for path integrals} \\ k=1: \text{Green's theorem in } \mathbb{R}^2,\ \text{curl in } \mathbb{R}^3 \\ k=2 \text{ in } \mathbb{R}^3: \text{Gauss / divergence thm.} \end{array}\right)$$

The most useful case for cx analysis is: $D\subset\mathbb{R}^2 \Rightarrow \displaystyle\int_{\partial D} p\,dx + q\,dy = \int_D\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dx\wedge dy$.

<u>Sketch proof:</u>

- both sides obey pullback formula (using $\varphi^* d\omega = d(\varphi^*\omega)$, and $\partial\varphi(M)=\varphi(\partial M)$). so can do changes of coordinates / pullback by parametrization $D\xrightarrow{\varphi} M$.
- can decompose into pieces (either by writing ω as sum of forms with support contained in subsets that have a single parametrization, or by observing

that if $M = M_1 \cup M_2$, $M_1 \cap M_2 = N \subset \partial M_i$:  then ∂M_1 and ∂M_2 contain N with opposite orientations, and so

$$\int_M d\omega = \int_{M_1} d\omega + \int_{M_2} d\omega \quad \& \quad \int_{\partial M} \omega = \int_{\partial M_1} \omega + \int_{\partial M_2} \omega.$$

- over a k-cell, and considering each component of $\omega \in \Omega^{k-1}$ separately: eg.

$D = \prod_{i=1}^{k} [a_i, b_i] = D' \times [a_k, b_k]$: $\quad \omega = f\, dx_1 \wedge \dots \wedge dx_{k-1} \Rightarrow d\omega = (-1)^{k-1} \frac{\partial f}{\partial x_k} dx_1 \wedge \dots \wedge dx_{k-1} \wedge dx_k$

$$\int_D d\omega = \int_D (-1)^{k-1} \frac{\partial f}{\partial x_k} |dx| \overset{\text{iterated integral}}{=} \int_{D'} \left(\int_{a_k}^{b_k} (-1)^{k-1} \frac{\partial f}{\partial x_k} dx_k \right) dx_1 \dots dx_{k-1}$$

$$\underset{\text{fund. th. calc.}}{=} (-1)^{k-1} \int_{D'} \left(f(x_1 \dots x_{k-1}, b_k) - f(x_1 \dots x_{k-1}, a_k) \right) dx_1 \dots dx_{k-1}$$

$$= (-1)^{k-1} \left(\int_{D' \times \{b_k\}} \omega - \int_{D' \times \{a_k\}} \omega \right) = \int_{\partial D} \omega$$

using that $\int \omega$ vanishes on the other faces of D ($\perp (x_1 \dots x_{k-1})$-plane) and orientation convention for ∂D (which we didn't state but is designed to make this work).

Our next topic: Complex analysis (in 1 complex variable)

We'll study functions $f: U \underset{\text{open}}{\subset} \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$.

Writing $z = x + iy$, these are instances of functions $\mathbb{R}^2 \to \mathbb{R}^2$, and the notion of continuity is the same, but we introduce a different (more restrictive) notion of differentiability.

Def: The (complex) derivative of f at $z \in U$ (if it exists) is
$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad (\text{ie. } f(z+h) = f(z) + h f'(z) + o(|h|))$$

The catch is: this limit has to hold for $h \to 0$ in $\mathbb{C}$ ...

Ex:
- assume f only takes real values, $f(z) \in \mathbb{R}\ \forall z \in \mathbb{C}$ ... Then in the def^{n} the numerator is always real, so taking $h \to 0$ in $\mathbb{R}$ we get $f'(z) \in \mathbb{R}$, while taking h imaginary we get $f'(z) \in i\mathbb{R}$. So: the complex derivative of a function which takes real values either doesn't exist or is equal to 0 ...!
- in general, we can treat $f: U \to \mathbb{C}$ as a function of 2 real variables $x + iy$.

If $f'(z)$ exists then: taking $h \in \mathbb{R}$ we find $\partial f / \partial x = f'(z)$; $h \in i\mathbb{R}$: $\partial f / \partial y = i f'(z)$ $\Rightarrow$ necess. $\boxed{\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}}$

This is the Cauchy-Riemann eqn.

Def: We say $f: U \to \mathbb{C}$ is analytic if $f'(z)$ exists for all $z \in \mathbb{C}$.