Recall: $f: U \subset \mathbb{C} \to \mathbb{C}$ is analytic if the complex derivative $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists at every point of U ($\Leftrightarrow$ real differentiable, and solves Cauchy-Riemann eq^n $\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$).

* Ex: polynomials, rational functions $\frac{P(z)}{Q(z)}$
* Main class of examples: power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (centered at $z=0$)

(or similarly, $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ centered at z_0).

Recall the radius of convergence: $\frac{1}{R} = \limsup |a_n|^{1/n}$. $R \in [0,\infty]$

→ for $|z|<R$, the series converges (absolutely: $\sum |a_n||z|^n$ converges) by the root test,

$\limsup \left(|a_n z^n|^{1/n}\right) = \frac{|z|}{R} < 1 \Rightarrow$ comparison with geometric series

→ for $|z|>R$ the series diverges; for $|z|=R$ it depends...

→ convergence is uniform over smaller disc $\bar D_r = \{|z|\le r\}$ $\forall r<R$.

This is by the Weierstrass M-test: $\sup_{z\in \bar D_r} |a_n z^n| = |a_n| r^n$, $\sum |a_n| r^n$ converges $(r<R)$

$\Rightarrow \sum a_n z^n$ converges uniformly on $\bar D_r$.

This is because of uniform Cauchy criterion for partial sums $s_n = \sum_{k=0}^{n} a_k z^k$:

for $n>m\ge N$, $\sup_{z\in\bar D_r} |s_n(z)-s_m(z)| = \sup_{\bar D_r} \left|\sum_{m+1}^{n} a_k z^k\right| \le \sum_{m+1}^{n} |a_k| r^k \le \sum_{N+1}^{\infty} |a_k| r^k \xrightarrow[\text{as } N\to\infty]{} 0$

→ hence $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is continuous over $D_R = \{|z|<R\}$.

→ the series $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence as f;

the partial sums $s_n(z)$ are analytic, $\left.\begin{matrix} s_n \to f \\ s_n' \to g \end{matrix}\right\}$ uniformly on $\bar D_r$ $\forall r<R$

$\Rightarrow$ Thm: $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on D_R and $f'(z) = g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

Pf: We work on the smaller disk D_r $(r<R)$ where uniform convergence holds; $D_R = \bigcup_{r<R} D_r$

we've already seen that, for real f^{ns} of 1 real variable, $\left.\begin{matrix} s_n \to f \\ s_n' \to g \end{matrix}\right\}$ uniformly $\Rightarrow f'=g$.

Unfortunately the proof used mean value thm., which doesn't hold here. But for power series there's an easier proof using mean value inequalities, thanks to... bounds on s_n'', which also converges uniformly on $\bar D_r$ hence $\exists$ uniform bound $|s_n''(z)| \le M$ $\forall n\in\mathbb{N}$ $\forall z\in\bar D_r$.

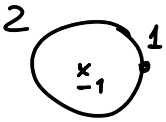
So: for $z, z+h \in D_r$, mean value inequalities (for $s_n(z+th)$, $t\in[0,1]$)

imply $|s_n(z+h) - s_n(z) - s_n'(z)h| \le \frac{1}{2}M|h|^2$.

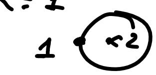
Taking limit as $n\to\infty$ we get $|f(z+h) - f(z) - g(z)h| \le \frac{1}{2}M|h|^2 \rightsquigarrow f'(z) = g(z)$. □

Ex: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ has $R=1$. For $|z|=1$ the series is always divergent (the terms don't $\to 0$), but the right hand side makes sense as soon as $z \neq 1$.

There are in fact expressions as power series over any disc not containing the pole $z=1$. Eg, around $z_0=-1$: $\frac{1}{1-z} = \frac{1}{2-(z+1)} = \sum_{n=0}^{\infty} \frac{(z+1)^n}{2^{n+1}}$ $R=2$



around $z_0 = 2$: $\frac{1}{1-z} = \frac{-1}{1+(z-2)} = \sum_{n=0}^{\infty} (-1)^{n+1}(z-2)^n$ $R=1$



(or even around ∞: $\frac{1}{1-z} = \frac{-1/z}{1-1/z} = -\sum_{n=1}^{\infty} (1/z)^n$)

Starting from $\sum z^n$, this process of extending past the disc of convergence is called analytic continuation; here it yields a rational function defined on $\mathbb{C} - \{1\}$.

Similarly for all rational functions! (on $\mathbb{C} - \{\text{poles}\}$). (eg. use partial fractions + case of $\frac{1}{(z-a)^k}$).

Ex: The partition generating function $\sum p(n) z^n$

$p(n) = \#$ partitions of n $= \#$ ways of writing n as a sum of positive integers

$(p(0) = 1)$. $= \# \{(a_k) \,/\, a_k \in \mathbb{N},\ \sum k a_k = n\}$ ($a_k = \#$ times k appears).

$$\Rightarrow f(z) = \sum p(n) z^n = (1+z+z^2+\dots)(1+z^2+z^4+\dots)(1+z^3+z^6+\dots)\dots$$
$$= \prod_{k=1}^{\infty} \frac{1}{1-z^k}$$

The series converges for $|z|<1$, and since there are manifestly poles at all complex roots of unity, we can't extend it past.

Ex: $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $R = \infty$, converges $\forall z \in \mathbb{C}$.

By algebraic manipulations, $\exp(z+w) = \exp(z)\exp(w)$. In particular $e^{-z} = \frac{1}{e^z}$
(remember: can multiply absolutely convergent series).

$e^z \neq 0 \ \forall z \in \mathbb{C}$

$e^{x+iy} = e^x e^{iy}$ has $|\cdot| = e^x$ and $\arg = y$.

- Define $\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$, $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, ...
  and usual properties follow (watch out if $z \notin \mathbb{R}$! $\cos(iy) = \cosh(y) \dots$).
- $\exp'(z) = \exp(z) \neq 0$ so $\exp$ is a local diffeomorphism near each point!
  Globally, $\exp: \mathbb{C} \to \mathbb{C}^*$ is the universal covering map!

What about logarithm? For $w \in \mathbb{C}^*$, want to define $\log(w) = z$ st. $e^z = w$.
Such z exists, but isn't unique: can add integer multiples of $2\pi i$.
$\mathrm{Re}\,\log(w)$ is well defined though, and equal to $\log|w|$ (for usual log on $\mathbb{R}_+$).

In general "$\log(w) = \log|w| + i\arg(w)$" not well defd & continuous on $\mathbb{C}^*$;
but ok over simply connected subsets of $\mathbb{C}^*$ (so can't go around $0 \Rightarrow \arg$ well defined).

This is consistent with what we've seen about lifting problem for

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ U \arrow[r, "i"'] \arrow[ur, dashed] & \mathbb{C}^* \end{tikzcd}$$

The same issue comes up with defining z^a for $a \notin \mathbb{Z}$:
would like to define it as $z^a = \exp(a \log z)$, but this only works on suitable domains. Eg. $\sqrt{z}$ is multivalued ($\pm\sqrt{z}$) and we can't define a continuous function on a domain that encloses the origin.

There are still power series expressions away from origin. Eg:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots, \qquad \sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \dots \quad (R=1)$$

---

* Now we consider <u>path integrals</u> of <u>complex 1-forms</u> $\omega = f(z)\,dz$:
given a continuous function $f: U \to \mathbb{C}$ and a (piecewise) differentiable path $\gamma: [0,1] \to \mathbb{C}$,

$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$$

(or: pick points $z_i = \gamma(t_i)$ along the path, with $\operatorname{diam} \gamma([t_i, t_{i+1}]) < \varepsilon$, then $\int = \lim\limits_{\varepsilon \to 0} \sum\limits_i f(z_i)(z_{i+1} - z_i)$)

<u>Ex</u>: $a \xrightarrow{\gamma} b$ $\quad \int_\gamma z^n\,dz = \int_0^1 \gamma(t)^n \gamma'(t)\,dt = \frac{1}{n+1}(b^{n+1} - a^{n+1})$

$\to$ for a power series $f(z) = \sum a_n z^n$, if γ is entirely contained in the disc of convergence, it follows that $\int_\gamma f(z)\,dz = F(b) - F(a)$, where $F(z) = \sum \frac{a_n}{n+1} z^{n+1}$: indeed $F' = f$ and so the equality follows from fundamental thm of calculus.

In general, a 1-form on $\mathbb{R}^2$ need not be exact & their path integrals need not be path-independent. One of the miracles is that things are much simpler in the analytic setting:

<u>Key result</u>: <u>Cauchy's theorem</u>:

> $D \subset \mathbb{C}$ bounded region with piecewise smooth boundary, $f(z)$ analytic on U open $\supset \overline{D}$
> Then $\int_{\partial D} f(z)\,dz = 0$.

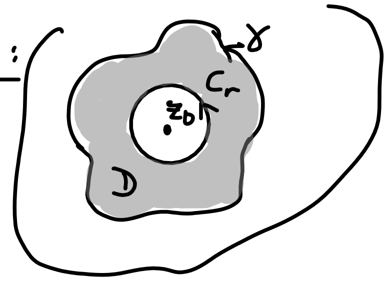
<u>Proof</u> <u>assuming f' is continuous</u>: the 1-form $\omega = f(z)\,dz$ is C^1, and
$d\omega = df \wedge dz = f'(z)\,dz \wedge dz = 0$. Stokes thm $\Rightarrow \int_{\partial D} \omega = \int_D d\omega = 0$. □

↳ let's check this more carefully, just to be safe:
($\omega = f(z)\,dz = f(z)\,dx + i f(z)\,dy \Rightarrow d\omega = \left(-\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial x}\right) dx \wedge dy = 0$ by Cauchy-Riemann)

We'll see later how to show that f analytic $\Rightarrow f'$ continuous. In the meantime we add the continuity of f' to our working assumptions.

* This holds not just for a simply connected region bounded by a simple closed curve!
We can also allow holes in the region D, eg. around points where f isn't defined.

Ex:



f analytic on $U-\{z_0\}$, γ enclosing z_0 as shown

$$\Rightarrow \int_\gamma f(z)\,dz = \int_{C_r = S^1(z_0,r)} f(z)\,dz.$$

(by Cauchy's theorem: $\partial D = \gamma - C_r$.)

* Now assume f is analytic on $U-\{z_0\}$ and $\lim_{z\to z_0} (z-z_0) f(z) = 0$.

(eg enough for f to be bounded near z_0).

Then $\left|\int_{C_r} f(z)\,dz\right| \leq \sup_{z\in C_r} |f(z)| \cdot \text{length}(C_r) = 2\pi r \sup_{z\in C_r} |f(z)| = 2\pi \sup_{z\in C_r} |(z-z_0) f(z)|$

Since this quantity $\to 0$ as $r\to 0$, and the path integral is independent of r, we get:

Thm ("improved Cauchy"): Cauchy's theorem $\left(\int_{\partial D} f(z)\,dz = 0\right)$ remains true under weaker assumption that f is defined & analytic in $D-\{z_0\}$, $z_0 \in \text{int}(D)$, and $\lim_{z\to z_0} (z-z_0) f(z) = 0$.

- However, we can't get rid of all assumptions about the behavior of f at z_0.

Example: $$\int_{S^1(z_0,r)} (z-z_0)^n\,dz = \int_0^{2\pi} (re^{i\theta})^n\, ire^{i\theta}\,d\theta = \begin{cases} 0 & \text{if } n\neq -1 \\ 2\pi i & \text{if } n=-1. \end{cases}$$

or any γ going once around z_0, by Cauchy! $\qquad z = z_0 + re^{i\theta}$

(cf. fundamental thm. / multivalued nature of log)

Using this, we get to Cauchy's integral formula:

Thm: $D\subset\mathbb{C}$ bounded region with piecewise smooth boundary γ, $f(z)$ analytic on an open domain containing $\overline{D}$, $z_0\in\text{int}(D)$ $\Rightarrow$ then

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-z_0}.} \quad (*)$$

Proof: • since $\int_\gamma \frac{dz}{z-z_0} = 2\pi i$, the formula is equivalent to:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0.$$

• The differentiability of f at z_0 implies: as $z\to z_0$, $\frac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)$,

and in particular $(z-z_0)\frac{f(z)-f(z_0)}{z-z_0} \to 0$. (+ analytic for $z\neq z_0$).

The result thus follows from improved Cauchy. $\square$

This is magical: the values of f at every point inside a closed curve γ can be determined by calculating path integrals on γ!! (assuming f defined and analytic everywhere in the enclosed region, of course). In this version, to emphasize we can vary the point of evaluation, one usually rewrites $(*)$ as: $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{w-z}$ $\forall z\in\text{int}(D)$, $\partial D = \gamma$

Next time we'll do even better: $\boxed{\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z)^{n+1}}}$

($\Rightarrow$ all derivatives exist!!)