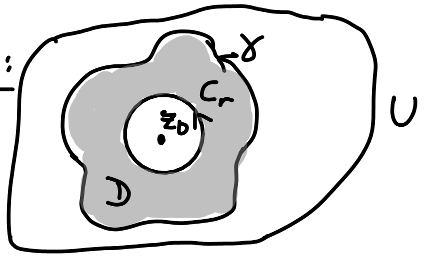
Math 55b Lecture 26 - Friday April 2

Cauchy's theorem: $D \subset \mathbb{C}$ bounded region with piecewise smooth boundary, $f(z)$ analytic on U open $\supset \overline{D}$: Then $\int_{\partial D} f(z)\,dz = 0$.

(proved so far under extra assumption that f' is continuous, via Stokes: $d(f(z)\,dz) = 0$).

Ex: 

f analytic on $U - \{z_0\}$, γ enclosing z_0 as shown

$$\Rightarrow \int_\gamma f(z)\,dz = \int_{C_r = S^1(z_0,r)} f(z)\,dz.$$

(by Cauchy's theorem: $\partial D = \gamma - C_r$.)

* Now assume f is analytic on $U - \{z_0\}$ and $\lim_{z\to z_0} (z-z_0) f(z) = 0$.

(eg this holds if f is bounded near z_0).

Then $\left|\int_{C_r} f(z)\,dz\right| \le \sup_{z\in C_r} |f(z)| \cdot \text{length}(C_r) = 2\pi r \sup_{z\in C_r} |f(z)| = 2\pi \sup_{z\in C_r} |(z-z_0) f(z)|$

Since this quantity $\to 0$ as $r \to 0$, and the path integral is independent of r, we get:

Thm: ("improved Cauchy") Cauchy's theorem $\left(\int_{\partial D} f(z)\,dz = 0\right)$ remains true under weaker assumption that f is defined & analytic in $D - \{z_0\}$, $z_0 \in \text{int}(D)$, and $\lim_{z\to z_0} (z-z_0) f(z) = 0$.

- However, we can't get rid of all assumptions about the behavior of f at z_0.

Example: $\int_{S^1(z_0,r)} (z-z_0)^n\,dz = \int_0^{2\pi} (re^{i\theta})^n\, ire^{i\theta}\,d\theta = \begin{cases} 0 & \text{if } n \ne -1 \\ 2\pi i & \text{if } n = -1. \end{cases}$

or any γ going once around z_0, by Cauchy! $z = z_0 + re^{i\theta}$ (cf. fundamental thm. / multivalued nature of log)

Using this, we get to Cauchy's integral formula:

Thm: $D \subset \mathbb{C}$ bounded region with piecewise smooth boundary γ, $f(z)$ analytic on an open domain containing $\overline{D}$, $z_0 \in \text{int}(D)$ $\Rightarrow$ then

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-z_0}.}$$

Proof: • since $\int_\gamma \frac{dz}{z-z_0} = 2\pi i$, the formula is equivalent to:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0.$$

• The differentiability of f at z_0 implies: as $z \to z_0$, $\frac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)$,

and in particular $(z-z_0)\frac{f(z)-f(z_0)}{z-z_0} \to 0$. (+ analytic for $z \ne z_0$).

The result thus follows from improved Cauchy. □

Alt. proof: Cauchy's thm gives $\frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi i}\int_{S^1(z_0,r)} \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta \xrightarrow[r\to 0]{} f(z_0)$. □

This is magical: the values of f at <u>every point</u> inside a closed curve γ can be determined by calculating path integrals on γ!! (assuming f defined and analytic everywhere in the enclosed region, of course). In this version, to emphasize we can vary the point of evaluation, one usually rewrites (*) as: $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{w-z}$

$\forall z \in \mathrm{int}(D)$, $\partial D = \gamma$

Even better: $\boxed{\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z)^{n+1}}}$ ($\Rightarrow$ all derivatives exist !!)

<u>Remark</u>: if f is given by a power series near z_0, $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ with $a_k = \frac{f^{(k)}(z_0)}{k!}$, then for $\gamma = S^1(z_0,r)$ small circle ($r <$ radius of convergence), uniform convergence of the series implies

$$\frac{1}{2\pi i}\int_{S^1(z_0,r)} \frac{f(w)\,dw}{(w-z_0)^{n+1}} = \sum_{k=0}^{\infty} \frac{a_k}{2\pi i}\underbrace{\int_{S^1(z_0,r)} \frac{(w-z_0)^k}{(w-z_0)^{n+1}}\,dw}_{\text{calc.} = \begin{cases}0 & \text{for } k\neq n\\ 2\pi i & k=n\end{cases}} = a_n = \frac{f^{(n)}(z_0)}{n!} \checkmark$$

+ Cauchy implies $\int_\gamma = \int_{S^1(z_0,r)}$.

But the problem is ... we haven't shown yet that analytic functions are power series! in fact the proof uses Cauchy's formula... so instead we have to work.

<u>Prop</u>: Suppose $\varphi(w)$ is continuous on $\gamma = \partial D$. Then $\forall n \geq 1$, $g_n(z) = \int_\gamma \frac{\varphi(w)\,dw}{(w-z)^n}$ is analytic in the interior of D, and $g_n'(z) = n\int_\gamma \frac{\varphi(w)\,dw}{(w-z)^{n+1}} = n g_{n+1}(z)$.

<u>Proof</u>: We first prove that g_n is continuous on $\mathrm{int}(D)$.

fix $z_0 \in \mathrm{int}(D)$, with $B_{2\delta}(z_0) \subset D$, and let $z \in B_\delta(z_0)$ (so z and z_0 are farther than δ away from all points of γ). Calculate:

$$\frac{1}{(w-z)^n} - \frac{1}{(w-z_0)^n} = \sum_{k=1}^{n} \frac{1}{(w-z)^{n-k}(w-z_0)^{k-1}}\left(\frac{1}{w-z} - \frac{1}{w-z_0}\right) = \sum_{k=1}^{n} \frac{z-z_0}{(w-z)^{n+1-k}(w-z_0)^k}$$

So:
$$g_n(z) - g_n(z_0) = \int_\gamma \varphi(w)\left(\frac{1}{(w-z)^n} - \frac{1}{(w-z_0)^n}\right)dw$$
$$= (z-z_0)\int_\gamma \varphi(w)\left(\sum_{k=1}^{n} \frac{1}{(w-z)^{n+1-k}(w-z_0)^k}\right)dz$$

Since each term in the sum has $|\cdot| \leq \frac{1}{\delta^{n+1}}$, this implies

$$\Rightarrow |g_n(z) - g_n(z_0)| \leq |z-z_0|\cdot\left(\sup_{w\in\gamma}|\varphi(w)|\right)\cdot\frac{n}{\delta^{n+1}}\,\mathrm{length}(\gamma).$$

Taking $z \to z_0$ this inequality proves that g_n is continuous at z_0, ie. g_n is continuous on $\mathrm{int}(D)$. Moreover,
$$\frac{g_n(z) - g_n(z_0)}{z-z_0} = \sum_{k=1}^{n}\int_\gamma \frac{\varphi(w)}{(w-z)^{n+1-k}(w-z_0)^k}\,dw \quad (*)$$

The continuity result, now applied to $\frac{\varphi(w)}{(w-z_0)^k}$, shows that the terms in the rhs. are continuous functions of $z \in \operatorname{int}(D)$, hence the rhs. of (*) is continuous, and its limit at $z = z_0$ equals $n \int_\gamma \frac{\varphi(w)}{(w-z_0)^{n+1}}\, dw = n g_{n+1}(z_0)$.

This gives the existence of $g_n'(z_0) = \lim_{z \to z_0} \frac{g_n(z) - g_n(z_0)}{z - z_0} = \lim_{z \to z_0} (\text{rhs of } (*)) = n g_{n+1}(z_0)$.

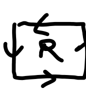
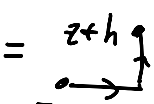
This holds $\forall z_0 \in \operatorname{int}(D)$, hence g_n is analytic as claimed and $g_n'(z) = n g_{n+1}(z)$. □

* Now if f is analytic in $U \supset \bar{D}$ then by Cauchy's integral formula, $2\pi i f(z) = \int_\gamma \frac{f(w)\, dw}{w - z}$ is the expression denoted $g_1(z)$ in the proposition, for $\varphi = f_{|\gamma}$.

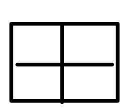
The proposition then shows that f is infinitely differentiable, all derivatives are analytic, and $2\pi i f^{(n)}(z) = n!\, g_{n+1}(z)$, ie. $\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(w)\, dw}{(w-z)^{n+1}}$. □.

* This also lets us lift the extra assumption we've made so far in all proofs using Cauchy's theorem, that f' is continuous.

Prop: If f is analytic then f' is continuous.

Pf: If f is analytic in a disc $D \ni z_0$, define $F(z) = \int_{z_0}^{z} f(w)\, dw$ where we choose a path consisting of horizontal & vertical line segments.
We don't have the full strength of Stokes' theorem (don't know f' continuous), but we claim it holds for rectangles:  $\int_{\partial R} f(w)\, dw = 0$. (see below).
Given this, our def^{n} of F makes sense & doesn't depend on path. We claim F is analytic and $F' = f$. Indeed: $F(z+h) - F(z) = \int_\gamma f(w)\, dw$ where $\gamma =$ 

Using continuity of f, as $h \to 0$ we have $\sup_{w \in \gamma} |f(w) - f(z)| \to 0$,
hence $F(z+h) - F(z) = h f(z) + o(|h|)$, hence $F'(z) = f(z)$.
So now F is analytic with continuous derivative $F' = f$, so we can apply Cauchy's integral formula and the above argument to F, so F has derivatives to all orders. In particular $F''(z) = f'(z)$ is continuous. □

<u>Cauchy's theorem on rectangles</u> (without assuming f' continuous):
Assume $R = R_0$ is a rectangle, f analytic, and $I = \int_{\partial R} f(z)\, dz \neq 0$
Cut R into 4 equal rectangles  then $\int_{\partial R} =$ sum of 4 path integrals, so

$\exists R_1 \subset R_0$ of $\mathrm{diam}(R_1) = \frac{\mathrm{diam}(R_0)}{2}$ st. $\left|\int_{\partial R_1} f(z)\,dz\right| \geq \frac{1}{4}|I|$. Repeat this process,

$R_0 \supset R_1 \supset R_2 \supset \dots$ with $\mathrm{diam}(R_n) = \frac{\mathrm{diam}(R_0)}{2^n}$ and $\left|\int_{\partial R_n} f(z)\,dz\right| \geq \frac{1}{4^n}|I|$.

$\bigcap_{n\in\mathbb{N}} R_n = \{z_0\}$ (a decreasing seq. of nonempty closed subsets in a compact space has a non-empty intersection: else complements would be an open cover w/out a finite subcover).

But now, $f(z) = f(z_0) + f'(z_0)(z-z_0) + r(z)$, $r(z) = o(|z-z_0|)$

$$\Rightarrow \left|\int_{\partial R_n} f(z)\,dz\right| = \left|\int_{\partial R_n} r(z)\,dz\right| \leq \mathrm{length}(\partial R_n) \cdot \sup_{\partial R_n} |r(z)| = o\left(\frac{1}{4^n}\right).$$ Contradiction. $\square$

Returning to Cauchy's integral formula for derivatives,

> $f(z)$ analytic on $U \subset \mathbb{C} \Rightarrow f$ has derivatives to all orders in U, all derivatives are analytic, and for $z \in \mathrm{int}(D) \subset \bar{D} \subset U$, $\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)\,dw}{(w-z)^{n+1}}$.

From this we get, by bounding the integral in the r.h.s.

<u>Thm:</u> (<u>Cauchy's bound</u>)
> If f is analytic in $U \supset \overline{B_R(z_0)}$, then $\left|\frac{f^{(n)}(z_0)}{n!}\right| \leq \frac{1}{R^n} \sup_{w\in S^1(z,R)} |f(w)|$.

(By considering $r<R$, $r \to R$, the result still holds under the weaker assumption that f is continuous on $\overline{B_R(z_0)}$ and analytic in $B_R(z_0)$).

* Cauchy's bound has important consequences for <u>entire functions</u>, ie. analytic on all of $\mathbb{C}$.

<u>Corollary:</u>
> If f is analytic on all of $\mathbb{C}$ ("entire function") and bounded, then f is constant.

(apply Cauchy's bound with $R\to\infty$ to get $f'=0$.)

<u>Corollary:</u>
> A nonconstant entire function $f:\mathbb{C}\to\mathbb{C}$ has dense image $\overline{f(\mathbb{C})} = \mathbb{C}$.

<u>Pf:</u> if $c \notin \overline{f(\mathbb{C})}$, then $\exists\,\varepsilon>0$ st. $|f(z)-c| \geq \varepsilon$ $\forall z\in\mathbb{C}$, and then $\frac{1}{f(z)-c}$ is a bounded entire function hence constant. $\square$

* There are even more important consequences for <u>Taylor series</u> of analytic functions.

<u>Corollary:</u>
> The power series $\sum_{n\geq 0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ (= the Taylor series of f at z_0) has radius of convergence $\geq R$, if f is analytic in $B_R(z_0)$.

(since Cauchy's bound implies $\left|\frac{f^{(n)}(z_0)}{n!}\right|^{1/n} \leq \frac{C(r)^{1/n}}{r}$ $\forall r<R$, so $\limsup \leq \frac{1}{r} \Rightarrow \leq \frac{1}{R}$)