Last time: Cauchy's integral formula for derivatives

| $f(z)$ analytic on $U\subset\mathbb{C}$ $\Rightarrow$ f has derivatives to all orders in U, all derivatives are analytic, and for $z\in \mathrm{int}(D)\subset\bar{D}\subset U$, $\dfrac{f^{(n)}(z)}{n!}=\dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\frac{f(w)\,dw}{(w-z)^{n+1}}$.

We now explore consequences of this formula.

First we get, by bounding the integral in the r.h.s:

Thm: (Cauchy's bound) || If f is analytic in $U\supset\overline{B_R(z_0)}$, then $\left|\dfrac{f^{(n)}(z_0)}{n!}\right|\le\dfrac{1}{R^n}\sup_{w\in S^1(z,R)}|f(w)|$.

(By considering $r<R$, $r\to R$, the result still holds under the weaker assumption that f is continuous on $\overline{B_R(z_0)}$ and analytic in $B_R(z_0)$).

* Cauchy's bound has important consequences for entire functions, ie. analytic on all of $\mathbb{C}$.

Corollary: || If f is analytic on all of $\mathbb{C}$ ("entire function") and bounded, then f is constant.

(apply Cauchy's bound with $R\to\infty$ to get $f'=0$.)

Corollary: || A nonconstant entire function $f:\mathbb{C}\to\mathbb{C}$ has dense image $\overline{f(\mathbb{C})}=\mathbb{C}$.

Pf: if $c\notin\overline{f(\mathbb{C})}$, then $\exists\,\varepsilon>0$ st. $|f(z)-c|\ge\varepsilon$ $\forall z\in\mathbb{C}$, and then $\dfrac{1}{f(z)-c}$ is a bounded entire function hence constant. $\square$

* There are even more important consequences for Taylor series of analytic functions.

Corollary: || The power series $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ (= the Taylor series of f at z_0) has radius of convergence $\ge R$, if f is analytic in $B_R(z_0)$.

(since Cauchy's bound implies $\left|\dfrac{f^{(n)}(z_0)}{n!}\right|^{1/n}\le\dfrac{C(r)^{1/n}}{r}$ $\forall r<R$, so $\limsup\le\frac{1}{r}\Rightarrow\le\frac{1}{R}$)

Theorem: $\boxed{\text{If } f \text{ is analytic in } B_R(z_0) \text{ then } f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n,\ a_n=\frac{f^{(n)}(z_0)}{n!},\ \text{over } B_R(z_0).}$

Pf: By change of variables $z-z_0$, we assume $z_0=0$. We prove the equality over slightly smaller discs $B_r=\{|z|<r\}$ $\forall r<R$; the Taylor series converges by the previous Corollary. For $z\in B_r$, write $f(z)=\dfrac{1}{2\pi i}\displaystyle\int_{S^1(r)}\frac{f(w)\,dw}{w-z}$ (Cauchy)

and note $\dfrac{1}{w-z}=\dfrac{1}{w(1-z/w)}=\dfrac{1}{w}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n$.

For fixed $z \in B_r$ this series of functions of $w \in S^1(r)$ converges uniformly (Weierstrass M-test, $\sum \left(\frac{|z|}{r}\right)^n$ converges since $|z|<r$).

So $$\frac{1}{2\pi i}\int_{S^1(r)} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{S^1(r)} \frac{f(w)\,z^n}{w^{n+1}}\,dw \underset{\substack{\uparrow\\ \text{Cauchy integral formula}}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,z^n. \qquad \square$$

Corollary: If $f(z) = \sum a_n z^n$ has radius of convergence R, then it has a singularity (where it cannot be analytically continued) on the circle $\{|z| = R\}$.

(because: if $\exists$ analytic function extending f on an open $U \supset \overline{B_R(0)}$, then $\exists \varepsilon > 0$ st. $B_{R+\varepsilon}(0) \subset U$, and then the radius of convergence would be $\geq R+\varepsilon$).

- Zeroes of analytic functions:

Corollary: If $f : U \to \mathbb{C}$ is analytic, $z_0 \in U$, if $f^{(n)}(z_0) = 0\ \forall n$ then $f(z) = 0$ on U.
connected open

Similarly: $f^{(n)}(z_0) = g^{(n)}(z_0)\ \forall n \Rightarrow f = g$ on U.

Pf: Let $V = \{z \in U / f^{(n)}(z) = 0\ \forall n\}$. The result on Taylor series implies: if $z \in V$ and $B_r(z) \subset U$ then over $B_r(z)$, f equals its Taylor series $\equiv 0$, and so $f^{(n)} = 0\ \forall n$ over $B_r(z)$. Hence V is open.
But $W = \{z \in U / \exists n\ f^{(n)}(z) \neq 0\} = \bigcup_{n \geq 0} \{z \in U / f^{(n)}(z) \neq 0\}$ is open too, and $U = V \sqcup W$. Since U is connected and $V \neq \emptyset$, $U = V$, so $f = 0$. $\square$

The key point here is: at a point where $f(z_0) = 0$, f vanishes to a finite integer order (unless $f \equiv 0$) – not to infinite or to fractional order as can happen with real functions.

Corollary: $f : U \to \mathbb{C}$ analytic, not everywhere zero, then the zeros of f are isolated.
connected open
(i.e. $f^{-1}(0)$ has no limit points).

Pf: if $f(z_0) = 0$ then writing $f(z) = \sum a_n (z-z_0)^n$, not all a_n are zero.
Let $k = \min\{n / a_n \neq 0\}$ (first nonzero term), $f(z) = (z-z_0)^k g(z)$ where $g(z) = \sum_{n \geq 0} a_{k+n}(z-z_0)^n$ is analytic over $B(z_0, R) \subset U$, and $g(z_0) = a_k \neq 0$.
By continuity, $\exists \varepsilon > 0$ st. $|z-z_0| < \varepsilon \Rightarrow g(z) \neq 0$
hence $0 < |z-z_0| < \varepsilon \Rightarrow f(z) \neq 0$: z_0 is an isolated point of $f^{-1}(0)$. $\square$

Non-example: $f(z) = \exp\left(\frac{2\pi i}{z}\right)$ satisfies $f\left(\frac{1}{n}\right) = 0\ \forall n \geq 1$ integer, but $\{\frac{1}{n} / n \in \mathbb{Z}_+\}$ has no limit points in the domain of f, $U = \mathbb{C}^*$.

Remark: in the real C^∞ world, there are nonzero functions with nonisolated zeroes.
(e.g. $f(x) = \exp\left(-\frac{1}{x^2}\right)\cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$).

Corollary (<u>uniqueness of analytic continuation</u>): $f, g : U \to \mathbb{C}$ analytic (U connected open), if $f = g$ on a nonempty open subset of U, or any subset with a limit point in U, then $f = g$ on U.

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* Some other consequences of Cauchy formula, for space of analytic functions w/ uniform topology

<u>Theorem</u>: If $f_n(z)$ are analytic functions on U, and $f_n \to f$ <u>locally uniformly</u> (i.e. $\forall z \in U\ \exists r > 0$ st. $\overline{B_r(z)} \subset U$ and $f_n \to f$ uniformly on $\overline{B_r(z)}$) ($\Leftrightarrow f_n \to f$ uniformly on all compact subsets of U) then f is analytic on U.

(This is very different from the real case: a sequence of C^∞ functions can converge uniformly to a nondifferentiable limit!)

<u>Proof</u>: Given a closed disk (or other compact) $B \subset U$ over which $f_n \to f$ uniformly, and $z \in \mathrm{int}(B)$, we have

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B} \frac{f_n(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\partial B}\frac{f(w)\,dw}{w-z}$$

Cauchy formula: f_n analytic

using $f_n \to f$ uniformly on ∂B, so $\displaystyle\int_{\partial B} \frac{f_n(w)-f(w)}{w-z}\,dw \xrightarrow[n\to\infty]{} 0$

Last time we saw a proposition stating: given that f is continuous on ∂B (which follows from uniform convergence), the rhs of this formula defines an analytic function on $\mathrm{int}(B)$. Thus f is analytic in $\mathrm{int}(B)$, hence on all U.

Even better:

<u>Thm</u>: If f_n analytic on U & converge locally uniformly to f, then f_n' converge locally uniformly to f', and so on for higher derivatives.

(<u>Pf</u>: same ingredients: Cauchy formula + uniform convergence).

So: analytic functions are a <u>closed</u> subspace of $C^0(U, \mathbb{C})$ with $C^0_{(loc)}$ topology of (local) uniform convergence, and moreover the $C^0_{loc}, C^1_{loc}, C^2_{loc} \dots$ topologies all coincide when we restrict them to the subspace of analytic functions (whereas in real analysis C^1 is strictly finer than C^0, etc.).

And we have a (sequential) compactness property too...

<u>Thm</u>: Any uniformly bounded sequence of analytic functions f_n on U has a subsequence which converges uniformly on compact sets to an analytic g.

Proof: If $K \subset U$ is compact, recall $\exists r > 0$ st. $\mathrm{dist}(K, \partial U) > r$,

so $\forall z \in K$, $|f_n'(z)| = \left| \frac{1}{2\pi i} \int_{S^1(z,r)} \frac{f_n(w)}{(w-z)^2}\, dw \right| \leq \frac{1}{2\pi} \frac{\sup |f_n|}{r^2} \,\mathrm{length}(S^1(z,r))$

$\leq \frac{1}{2\pi r} \sup_K |f_n|$

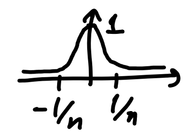
Since (f_n) is uniformly bounded this gives a uniform bound on $|f_n'|$ on K independently of n.

Hence f_n is uniformly equicontinuous on K ($\forall \varepsilon\ \exists \delta$ st. $\forall z\ \forall n \ldots$). indep. of n

$\Rightarrow$ by Ascoli-Arzela, $\exists$ subsequence of (f_n) which converges uniformly on K.

(We can ensure uniform convergence on all compacts by considering a sequence of compacts K_n with $\bigcup_n K_n = U$, eg. $K_n = \{z \,/\, |z| \leq n,\ d(z, U^c) \geq \frac{1}{n}\}$. and using a diagonal process to get a sub-sub-...-subsequence that converges uniformly on all of them.) $\square$

Ex: in real analysis, a standard example for a bounded sequence of continuous (C^∞) functions that isn't equicontinuous over $[-a,a]$ $\forall a > 0$ is $f_n(x) = \frac{1}{1+n^2x^2}$



(& has no uniformly convergent subseq., since pointwise limit $\notin C^0$).

These extend to analytic functions $f_n(z) = \frac{1}{1+n^2z^2}$, but the above theorem doesn't apply to these near 0 because f_n has a pole at $z = \pm i/n$, so the sequence isn't uniformly bounded on any fixed neighborhood of 0, and that's why equicontinuity fails over $\mathbb{R}$!

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* Besides the magical stuff (derivatives to all orders, Cauchy's formula, convergence of Taylor series) we also have more basic things that carry over from real analysis, eg. <u>antiderivatives</u> and inverse functions... but these come with caveats.

- **Thm:** If $f(z)$ is analytic on a simply connected open $U \subset \mathbb{C}$ then $\exists$ analytic function $F: U \to \mathbb{C}$ st. $F'(z) = f(z)$.

This is because we can define $F(z) = \int_{z_0}^{z} f(z)\, dz$, Cauchy's thm implies that the choice of path doesn't matter: given any piecewise differentiable closed loop γ in U, $\int_\gamma f(z)\, dz = 0$. In fact, over discs $B_r(z_0) \subset U$ we can define F by term-by-term integration of the power series expression for f.

Simply connected is <u>necessary</u>! eg. $f(z) = \frac{1}{z}$ on $\mathbb{C}^* = \mathbb{C} - \{0\}$, can only integrate to $F(z) = \log z$ over a simply connected subset (not allowing paths that enclose 0).

- **Thm:** If f is analytic near a, with $f(a) = b$ and $f'(a) \neq 0$, then $\exists$ analytic inverse function g defined on a neighborhood of b, st $g(b) = a$ & $g'(b) = 1/f'(a)$.

This is a direct consequence of the inverse function theorem for $f: \mathbb{R}^2 \to \mathbb{R}^2$, together with observation that $f'(a) \neq 0 \Rightarrow Df(a)$ is invertible, and its inverse is also complex-linear.

Rmk: for real functions of 1 real variable, can do this on any connected interval where $f' \neq 0$ ($\Rightarrow f$ injective), but in complex world this isn't true, even on simply connected domains - eg. $\left.\begin{array}{l}\log = \text{inverse function of } \exp, \\ \sqrt[n]{z} = \text{inverse function of } z^n\end{array}\right\}$ defined only on suitable domains.

The inverse function theorem does give: $\exp'(z) = e^z \Rightarrow \log'(z) = \frac{1}{z}$.

& derivative of $z^{1/n}$ is $\frac{1}{n} z^{-(n-1)/n}$.

from which we can get eg. power series expressions $\log(1+z) = \int \frac{dz}{1+z} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ $(R=1)$.

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \cdots$$

These have singularities at $z=0$ - "branch singularities", <u>not</u> poles.

We'll soon study the behavior of analytic functions at an isolated singularity, ie. st. f is defined on $U - \{z_0\}$, $z_0 \in \text{int}(U)$, but this <u>won't</u> handle $\log z$ or z^α which aren't analytic on a whole $D^*(r) = D(r) - \{0\}$.