* Last time we saw several surprising consequences of Cauchy's formula (derivatives to all orders, bounds on them, convergence of Taylor series, isolated zeroes, unique continuation, ...).

We finished with a result about space of analytic functions $U \to \mathbb{C}$ with the C^0_{loc} topology of uniform convergence on compact subsets of U:

> If $f_n \to f$ in C^0_{loc} (ie. uniformly on compact subsets of U) and f_n is analytic then f is analytic, and in fact $f_n' \to f'$ uniformly on compact subsets.

This generalizes statements we saw earlier about convergence, analyticity, and derivatives of power series. It says that analytic functions are a <u>closed</u> subspace of $C^0(U,\mathbb{C})$ with $C^0_{(loc)}$ topology of (local) uniform convergence, and moreover the $C^0_{loc}, C^1_{loc}, C^2_{loc}$ ... topologies all coincide when we restrict them to the subspace of analytic functions (whereas in real analysis C^1 is strictly finer than C^0, etc.).
(also contrast with Stone-Weierstrass, which says various classes of real functions are dense in C^0, hence very far from closed...)

And we have a (sequential) compactness property too...

<u>Thm:</u>
> Any uniformly bounded sequence of analytic functions f_n on U has a subsequence which converges uniformly on compact sets to an analytic g.

<u>Proof:</u> If $K \subset U$ is compact, recall $\exists r > 0$ st. $\mathrm{dist}(K, \partial U) > r$,

so $\forall z \in K$, $|f_n'(z)| = \left| \frac{1}{2\pi i} \int_{S^1(z,r)} \frac{f_n(w)}{(w-z)^2}\, dw \right| \le \frac{1}{2\pi} \frac{\sup |f_n|}{r^2} \mathrm{length}(S^1(z,r)) \le \frac{1}{r} \sup_U |f_n|$

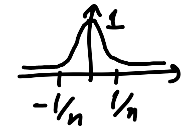
Since (f_n) is uniformly bounded this gives a uniform bound on $|f_n'|$ on K independently of n. (cf. Cauchy's bound!)

Hence f_n is uniformly <u>equicontinuous</u> on K ($\forall \varepsilon\ \exists \delta$ st. $\forall z\ \forall n$ ...). (indep. of n)

$\Rightarrow$ by Ascoli-Arzela, $\exists$ subsequence of (f_n) which converges uniformly on K.

(We can ensure uniform convergence on all compacts by considering a sequence of compacts K_n with $\bigcup_n K_n = U$, eg. $K_n = \{z \,/\, |z| \le n,\ d(z, U^c) \ge \frac{1}{n}\}$ and using a diagonal process to get a sub-sub-...-subsequence that converges uniformly on all of them.) □

<u>Ex:</u> in real analysis, a standard example for a bounded sequence of continuous (C^∞) functions that isn't equicontinuous over $[-a,a]$ $\forall a > 0$ is $f_n(x) = \frac{1}{1+n^2x^2}$



(& has no uniformly convergent subseq., since pointwise limit $\notin C^0$).

These extend to analytic functions $f_n(z) = \frac{1}{1+n^2z^2}$, but the above theorem doesn't apply to these near 0 because f_n has a pole at $z = \pm i/n$, so the sequence isn't uniformly bounded on any fixed neighborhood of 0, and that's why equicontinuity fails over $\mathbb{R}$!

We also have more basic things that carry over from single variable real analysis, such as <u>antiderivatives</u> and inverse functions... but these come with caveats.

- <u>Thm:</u> If $f(z)$ is analytic on a simply connected open $U \subset \mathbb{C}$ then $\exists$ analytic function $F: U \to \mathbb{C}$ st. $F'(z) = f(z)$.

This is because we can define $F(z) = \int_{z_0}^{z} f(z)\,dz$, Cauchy's thm implies that the choice of path doesn't matter: given any piecewise differentiable closed loop γ in U, $\int_\gamma f(z)\,dz = 0$. In fact, over discs $B_r(z_0) \subset U$ we can define F by term-by-term integration of the power series expression for f.

Simply connected is <u>necessary</u>! eg. $f(z) = \frac{1}{z}$ on $\mathbb{C}^* = \mathbb{C} - \{0\}$, can only integrate to $F(z) = \log z$ over a simply connected subset (not allowing paths that enclose 0).

- <u>Thm:</u> If f is analytic near a, with $f(a) = b$ and $f'(a) \neq 0$, then $\exists$ analytic inverse function g defined on a neighborhood of b, st $g(b) = a$ & $g'(b) = 1/f'(a)$.

This is a direct consequence of the inverse function theorem for $f: \mathbb{R}^2 \to \mathbb{R}^2$, together with observation that $f'(a) \neq 0 \Rightarrow Df(a)$ is invertible, and its inverse is also complex-linear.

<u>Rmk:</u> for real functions of 1 real variable, can do this on any connected interval where $f' \neq 0$ ($\Rightarrow f$ injective), but in complex world this isn't true, even on simply connected domains – eg. $\left.\begin{array}{l}\log = \text{inverse function of } \exp, \\ \sqrt[n]{z} = \text{inverse function of } z^n\end{array}\right\}$ defined only on suitable domains.

The inverse function theorem does give: $\exp'(z) = e^z \Rightarrow \log'(z) = \frac{1}{z}$.
& derivative of $z^{1/n}$ is $\frac{1}{n} z^{-(n-1)/n}$.

from which we can get eg. power series expressions $\log(1+z) = \int \frac{dz}{1+z} = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$ $(R=1)$.

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \dots$$

These have singularities at $z=0$ – "branch singularities", <u>not</u> poles.

We'll now study the behavior of analytic functions at an isolated singularity, ie. st. f is defined on $U - \{z_0\}$, $z_0 \in \mathrm{int}(U)$, but this <u>won't</u> handle $\log z$ or z^α which aren't analytic on a whole $D^*(r) = D(r) - \{0\}$.

<u>Laurent series</u>: these are power series with positive <u>and</u> negative exponents!

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n .$$ Convergence is best understood by splitting into

$\sum_{n\geq 0} a_n z^n$ (usual power series) converges for $|z| < R_2 = \dfrac{1}{\limsup_{n\to+\infty} |a_n|^{1/n}}$

and $\sum_{n<0} a_n z^n$ (power series in $\frac{1}{z}$) converges for $|z| > R_1 = \limsup_{n\to-\infty} |a_n|^{1/|n|}$

$\Rightarrow$ we have an annulus of convergence $\{R_1 < |z| < R_2\}$.

<u>Beware</u>: general (formal) Laurent series don't form a ring. The issue is that the coefficient of z^n in $(\sum a_k z^k)(\sum b_k z^k)$ should be $\sum_{k\in\mathbb{Z}} a_k b_{n-k}$, which may not be a convergent series. (Things are fine if annuli of convergence have non-empty intersection.) A better-behaved class of Laurent series are those with only finitely many negative powers of z, ie. $\sum_{-N}^{\infty} a_n z^n$ $\left(= \frac{1}{z^N}\cdot(\text{power series})\right)$.
These are actually a <u>field</u> - the field of fractions of the ring of power series.

<u>Thm</u>: If $f(z)$ is analytic in $A_{R_1,R_2} = \{R_1 < |z| < R_2\}$ then we can express it as a Laurent series $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ which converges on A_{R_1,R_2}.

<u>Pf</u>: We show this on slightly smaller annuli $\{r_1 \leq |z| \leq r_2\}$ $\forall R_1 < r_1 < r_2 < R_2$.
Then the Cauchy formula for A_{r_1,r_2} and its boundary $S^1(r_2) - S^1(r_1)$ gives

$$f(z) = \frac{1}{2\pi i}\int_{S^1(r_2)} \frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i}\int_{S^1(r_1)} \frac{f(w)\,dw}{w-z} \quad \text{for } r_1 < |z| < r_2 .$$

On $S^1(r_2)$ we have $(|z/w| < 1)$ $\dfrac{1}{w-z} = \dfrac{w^{-1}}{1-z/w} = \sum_{n=0}^{\infty} \dfrac{z^n}{w^{n+1}}$, converging uniformly

On $S^1(r_1)$ we have $(|w/z| < 1)$ $\dfrac{1}{z-w} = \dfrac{z^{-1}}{1-w/z} = \sum_{k=0}^{\infty} \dfrac{w^k}{z^{k+1}} = \sum_{n\leq -1} \dfrac{z^n}{w^{n+1}}$, $(n = -k-1)$ also converging uniformly

Uniform convergence allows us to move the sum outside of the integrals, giving

$$f(z) = \sum_{n\geq 0} \frac{1}{2\pi i} z^n \int_{S^1(r_2)} \frac{f(w)\,dw}{w^{n+1}} + \sum_{n\leq -1} \frac{1}{2\pi i} z^n \int_{S^1(r_1)} \frac{f(w)\,dw}{w^{n+1}} .$$

$$= \sum_{n\in\mathbb{Z}} a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\int_{S^1(r)} \frac{f(w)\,dw}{w^{n+1}}$$ (for any $r \in (R_1, R_2)$, since this is indep.^{t} of r by Cauchy). $\square$

(compare with our earlier result about Taylor series).

<u>Corollary</u>: any analytic function on $\{R_1 < |z| < R_2\}$ can be written as the sum of an analytic function on $\{|z| < R_2\}$ and an analytic function on $\{|z| > R_1\}$.

<u>Singularities and removability</u>: Assume f is analytic on $D^*(R) = D(R) - \{0\}$, and express it as a Laurent series $\sum_{n\in\mathbb{Z}} a_n z^n$. Let $N = \inf\{n \in \mathbb{Z} \,/\, a_n \neq 0\}$ (if exists)

1) If $N \geq 0$ (ie. $a_n = 0\ \forall n < 0$), f is a power series and the singularity at 0 is <u>removable</u>, ie. can extend f to an analytic function on $D(R) \ni 0$.
   - $N = \infty$ ie. $a_n = 0\ \forall n$: then $f \equiv 0$.
   - $N > 0$, then $f(z) = z^N(a_N + \dots)$ has an isolated zero of order N at 0.
   - $N = 0$, $f(0) = a_0 \neq 0$

The new cases are when the negative part of the Laurent series isn't zero.

2) If $N < 0$ finite, ie. there are finitely many negative powers of z in the series: then $f(z) = \dfrac{1}{z^{|N|}}(a_N + \dots) = \dfrac{g(z)}{z^{|N|}}$, g analytic with $g(0) = a_N \neq 0$.
   We say f has a <u>pole</u> of order $|N|$ at 0.

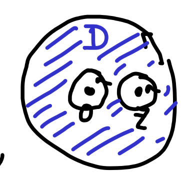
3) If $N = -\infty$, ie. the negative part of the series has ∞ many terms: we say f has an <u>essential singularity</u> at 0 (= non-removable singularity other than a pole).
   <u>Ex</u>: $\exp(1/z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$ essential singularity at 0.

The <u>qualitative differences between the 3 cases</u> can also be understood without involving Laurent series.

<u>Thm</u>: f analytic on $D^*(R)$:
1) the singularity at 0 is removable iff $f(z)$ is bounded on a neighborhood of 0.
2) f has a pole at 0 iff $|f(z)| \to \infty$ as $z \to 0$
3) f has an essential singularity iff $\forall \varepsilon > 0$, $f(D^*(\varepsilon))$ is dense in $\mathbb{C}$
   (equivalently: $\forall y \in \mathbb{C} \cup \{\infty\}$, $\exists z_n \to 0$ st. $f(z_n) \to y$).

<u>Pf</u> (without using Laurent series!)

1) assume f bounded on $\overline{D^*(r)}$. Since f is continuous on $S^1(r)$, we have seen that
$$g(z) = \frac{1}{2\pi i}\int_{S^1(r)} \frac{f(w)\,dw}{w - z}$$
is analytic in $D(r)$. By Cauchy's formula, if $0 < \varepsilon < \frac{|z|}{2}$, then
$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(w)\,dw}{w - z} = \frac{1}{2\pi i}\left(\int_{S^1(r)} - \int_{S^1(z,\varepsilon)} - \int_{S^1(0,\varepsilon)}\right)$$
$$= g(z) - f(z) - \frac{1}{2\pi i}\int_{S^1(0,\varepsilon)} \frac{f(w)\,dw}{w - z} = 0$$



but the last integral $\to 0$ as $\varepsilon \to 0$ since the integrand is bounded and $\text{length}(S^1(\varepsilon)) \to 0$.

So: g is analytic in $D(r)$ and $g(z) = f(z)$ $\forall z \in D(r) - \{0\}$.

ie. the singularity at 0 is removable.

(Conversely, it is clear that f is bounded near 0 if the sing. is removable).

2) assume $|f| \to \infty$ as $z \to 0$, then $h(z) = \dfrac{1}{f(z)}$ is analytic and bounded in a neighborhood of 0, hence has a removable singularity, ie. $\exists$ analytic extension which we denote again by h. Since $|h| \to 0$ as $z \to 0$, h has an (isolated) zero at $z = 0$, where it vanishes to finite order: $\exists n \geq 1$ and $k(z)$ analytic, $k(0) \neq 0$ st. $h(z) = z^n k(z)$.

Hence $f(z) = \dfrac{1}{h(z)} = \dfrac{g(z)}{z^n}$ where $g(z) = \dfrac{1}{k(z)}$ is analytic on a nbd. of 0: f has a pole of order n.

Conversely if $f(z) = \dfrac{g(z)}{z^n}$, $n \geq 1$, g analytic, $g(0) \neq 0$ then $\exists c > 0$ st. $|g(z)| \geq c > 0$ over a neighborhood of 0, and $|f(z)| \geq \dfrac{c}{|z|^n} \to \infty$ as $z \to 0$.

3) if $f(D^*(\varepsilon))$ isn't dense in $\mathbb{C}$, then $\exists c$ st. $h(z) = \dfrac{1}{f(z) - c}$ is bounded near 0, hence has a removable singularity; we denote the extension over 0 by h again.

If $h(0) = 0$ then, as in the previous case, h has a zero of finite order $n \geq 1$, $\dfrac{1}{h(z)}$ has a pole of order n, and $f(z) = c + \dfrac{1}{h(z)}$ also has a pole of order n.

If $h(0) \neq 0$ then $f(z) = c + \dfrac{1}{h(z)}$ extends over 0, the singularity is removable.

So: essential singularity $\Rightarrow$ $f(D^*(\varepsilon))$ is dense in $\mathbb{C}$ $\forall \varepsilon > 0$.

(the converse is clear too: $f(D^*(\varepsilon))$ dense $\Rightarrow$ f isn't bounded and $|f| \not\to \infty$, so neither removable nor pole).

$\square$