**Singularities and removability:** If $f$ is analytic on $D^*(R) = D(R) - \{0\}$, we can express it as a Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$.

1) If there are no negative powers of $z$ (ie. $a_n = 0\ \forall n<0$), $f$ is a power series and the singularity at $0$ is removable, ie. can extend $f$ to an analytic function on $D(R) \ni 0$, with $f(0) = a_0$. If $N = \min\{n / a_n \neq 0\} > 0$, then $f(z) = z^N(a_N + \dots)$ has a zero of order $N$ at $z=0$.

2) If there are finitely many negative powers of $z$ in the series: let $-N = \min\{n / a_n \neq 0\} < 0$

then $f(z) = \sum_{n=-N}^{\infty} a_n z^n = \frac{1}{z^N}(a_{-N} + \dots) = \frac{g(z)}{z^N}$, $g$ analytic with $g(0) = a_{-N} \neq 0$.

We say $f$ has a pole of order $N$ at $0$.

3) If the negative part of the series has $\infty$ many terms: we say $f$ has an essential singularity at $0$ (= non-removable singularity other than a pole).

Ex: $\exp(1/z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$ essential singularity at $0$.

The qualitative differences between the 3 cases can also be understood without involving Laurent series.

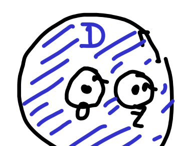
Thm: $f$ analytic on $D^*(R)$:
1) the singularity at $0$ is removable iff $f(z)$ is bounded on a neighborhood of $0$.
2) $f$ has a pole at $0$ iff $|f(z)| \to \infty$ as $z \to 0$
3) $f$ has an essential singularity iff $\forall \varepsilon > 0$, $f(D^*(\varepsilon))$ is dense in $\mathbb{C}$
(equivalently: $\forall y \in \mathbb{C} \cup \{\infty\}$, $\exists z_n \to 0$ st. $f(z_n) \to y$).

Pf (without using Laurent series!)

1) assume $f$ bounded on $\overline{D^*(r)}$. Since $f$ is continuous on $S^1(r)$, we have seen that

$g(z) = \frac{1}{2\pi i}\int_{S^1(r)} \frac{f(w)\,dw}{w-z}$ is analytic in $D(r)$. By Cauchy's formula, if

$0 < \varepsilon < \frac{|z|}{2}$, then $\frac{1}{2\pi i}\int_{\partial D} \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\left(\int_{S^1(r)} - \int_{S^1(z,\varepsilon)} - \int_{S^1(0,\varepsilon)}\right)$




$= g(z) - f(z) - \frac{1}{2\pi i}\int_{S^1(0,\varepsilon)} \frac{f(w)\,dw}{w-z} = 0$

but the last integral $\to 0$ as $\varepsilon \to 0$ since the integrand is bounded and $\text{length}(S^1(\varepsilon)) \to 0$.

So: $g$ is analytic in $D(r)$ and $g(z) = f(z)\ \forall z \in D(r) - \{0\}$.

ie. the singularity at $0$ is removable.

(Conversely, it is clear that $f$ is bounded near $0$ if the sing. is removable).

2) assume $|f| \to \infty$ as $z \to 0$, then $h(z) = \frac{1}{f(z)}$ is analytic and bounded in a neighborhood of 0, hence has a removable singularity, ie. $\exists$ analytic extension which we denote again by $h$. Since $|h| \to 0$ as $z \to 0$, $h$ has an (isolated) zero at $z=0$, where it vanishes to finite order: $\exists n \geq 1$ and $k(z)$ analytic, $k(0) \neq 0$ st. $h(z) = z^n k(z)$.

Hence $f(z) = \frac{1}{h(z)} = \frac{g(z)}{z^n}$ where $g(z) = \frac{1}{k(z)}$ is analytic on a nbd. of 0: $f$ has a pole of order $n$.

Conversely if $f(z) = \frac{g(z)}{z^n}$, $n \geq 1$, $g$ analytic, $g(0) \neq 0$ then $\exists c > 0$ st. $|g(z)| \geq c > 0$ over a neighborhood of 0, and $|f(z)| \geq \frac{c}{|z|^n} \to \infty$ as $z \to 0$.

3) if $f(D^*(\varepsilon))$ isn't dense in $\mathbb{C}$, then $\exists c$ st. $h(z) = \frac{1}{f(z) - c}$ is bounded near 0, hence has a removable singularity; we denote the extension over 0 by $h$ again.

If $h(0) = 0$ then, as in the previous case, $h$ has a zero of finite order $n \geq 1$, $\frac{1}{h(z)}$ has a pole of order $n$, and $f(z) = c + \frac{1}{h(z)}$ also has a pole of order $n$.

If $h(0) \neq 0$ then $f(z) = c + \frac{1}{h(z)}$ extends over 0, the singularity is removable.

So: essential singularity $\Rightarrow f(D^*(\varepsilon))$ is dense in $\mathbb{C}$ $\forall \varepsilon > 0$.

(the converse is clear too: $f(D^*(\varepsilon))$ dense $\Rightarrow f$ isn't bounded and $|f| \not\to \infty$, so neither removable nor pole). $\square$

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**Def:** If $f$ is analytic in $U \setminus \{p_1, \dots, p_n\}$ and has poles at $p_1 \dots p_n$ (no essential sings) then we say $f$ is <u>meromorphic</u> in $U$.

- If $f : U \setminus \{p_i\} \to \mathbb{C}$ is meromorphic with poles at $p_i$, then $|f(z)| \to \infty$ as $z \to p_i$, so $1/f$ has a removable singularity at $p_i$, where it has a zero (of order = pole order of $f$)

Hence $f$ extends to $\hat{f} : U \to S = \mathbb{C} \cup \{\infty\}$ Riemann sphere by setting $\hat{f}(p_i) = \infty$, and $\hat{f}$ is continuous and <u>analytic</u>, in the sense that

- away from the poles $\{p_i\} = \hat{f}^{-1}(\infty)$, $\hat{f}$ takes values in $\mathbb{C}$ and is analytic
- away from the zeros, $\frac{1}{\hat{f}(z)}$ takes values in $\mathbb{C}$ and is analytic (= analytic extension of $\frac{1}{f}$ over the removable sings. at $p_i$).

- These considerations tell us:
  - $\to$ zeros and poles of (non-identically zero) meromorphic functions are isolated.
  - $\to$ if $f$ and $g$ are analytic on $U$, $g \not\equiv 0$, then $f/g$ is meromorphic on $U$.
    if $f$ and $g$ have no common zeros, $f/g$ has zeros = zeros of $f$, poles = zeros of $g$.
    if there's a common zero, highest order wins (factor out powers of $(z - z_0)$).

→ another perspective on this: Laurent series with finite negative part are the field of fractions of power series ( $a_0+a_1z+\dots$ has an inverse in $\mathbb{C}[[z]]$ iff $a_0\neq 0$, and otherwise $(a_nz^n+\dots)^{-1} = \frac{1}{a_nz^n}(1+\dots)$ ), so a ratio of two nontrivial power series gives such a Laurent series hence defines a meromorphic function.

→ the converse is actually true (won't prove): every meromorphic function is a quotient $f/g$ of analytic functions. So: meromorphic functions are the field of fractions of the ring of analytic functions.

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- Assume $f$ is meromorphic on all $\mathbb{C}$ (ie. $f$ analytic on $\mathbb{C}-\{p_i\}$, w/ poles at $p_i$). If $|f(z)|$ is either bounded or $\to\infty$ as $|z|\to\infty$, then the function $g(w)=f(1/w)$ has a removable singularity or a pole at $w=0$, so it is meromorphic near $0$.
  $\Rightarrow$ can extend $\hat{f}$ to the Riemann sphere by setting $\hat{f}(\infty)=\hat{g}(0)$.
  Thus: if $f(z)$ and $f(\frac{1}{z})$ are meromorphic, get an analytic extension $\hat{f}: S\to S$ to the whole Riemann sphere!
- In fact, such $\hat{f}$ is necessarily a rational function. Indeed:

<u>Theorem</u>: If $f$ is an analytic entire function $f:\mathbb{C}\to\mathbb{C}$ and $|f(z)|\le M|z|^n$ near $|z|\to\infty$ then $f$ is a polynomial of degree at most $n$.

This follows from Cauchy's bound for derivative: $f^{(n)}(z)$ is a bounded entire function and hence constant (homework!!)

<u>Corollary</u>: If $f: S\to S$ is analytic (ie. $f(z)$ and $f(\frac{1}{z})$ are meromorphic) then $f$ is a rational function.

<u>Proof</u>: 
- the fact that $g(w)=f(\frac{1}{w})$ is meromorphic near $0$ gives a bound of the form
$$|g(w)|\le \frac{C}{|w|^n} \text{ for } w\to 0, \text{ ie. } |f(z)|\le C|z|^n \text{ for } z\in\mathbb{C},\ z\to\infty$$
- $f$ isn't an entire function, it does have poles - but only finitely many of them (poles of $f$ are zeros of $1/f$ hence isolated, and $S$ is compact).

so: $\exists$ polynomial $P(z)=\prod(z-p_i)^{n_i}$ ($p_i$ poles of $f$, $n_i$ order) st. $P(z)f(z)$ extends to an entire function on $\mathbb{C}$, also satisfying a bound $|P(z)f(z)|\le C'|z|^{n+\deg P}$ as $z\to\infty$. By the previous thm, $P(z)f(z)$ is a polynomial. □

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<u>Local behavior of analytic functions</u>: maximum principle and open mapping principle.

* Cauchy's integral formula can be viewed as a <u>mean value formula</u>:

<u>Thm</u>: If $f$ is analytic on $U\supset\overline{B_r(z)}$ then $f(z)$ is the average value of $f$ on $S^1(z,r)$.

Pf: by Cauchy, $f(z) = \frac{1}{2\pi i}\int_{S^1(z,r)} \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}}\, d(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta$. $\square$

- Corollary: the maximum principle

Thm: If $f$ is analytic on $U \subset \mathbb{C}$ (open connected) & nonconstant, then $|f|$ doesn't achieve its maximum anywhere in $U$. In particular, if $f$ is analytic on $U$ and continuous on $\bar{U}$, $\bar{U}$ compact, then the maximum of $|f|$ on $\bar{U}$ is achieved on the boundary of $U$.

Pf: Given $z_0 \in U$, $r>0$ small so $\bar{B}_r(z_0) \subset U$, we have $|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right| \underset{(*)}{\leq} \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta \leq \max_{S^1(z_0,r)} |f|$.

If $|f|$ has a (local) max. at $z_0$ then $\max_{S^1(z_0,r)} |f| = |f(z_0)|$ and these inequalities are equalities.

This implies $|f(z)| = |f(z_0)|$ $\forall z \in S^1(z_0,r)$. In fact $f(z) = f(z_0)$: if $\arg(f(z))$ varies then $(*)$ is $<$.

(eg. rescale so $f(z_0)=1$, then $|f(z)| \leq 1$ so $Re(f(z)) \leq 1$, and $Re\, f(z_0) = Re\left(\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right) \leq 1$, equality implies $Re(f(z)) = 1$ $\forall z \in S^1(z_0,r)$, and since $|f(z)| \leq 1$, this gives $f(z)=1$ $\forall z \in S^1(z_0,r)$).

Since $f$ is analytic, $f(z)-f(z_0) = 0$ $\forall z \in S^1(z_0,r) \Rightarrow$ zeros of $f(z)-f(z_0)$ aren't isolated (zeros of nontrivial analytic f$^{ns}$ are isolated) $\Rightarrow f(z)-f(z_0) = 0$ on $U \Rightarrow f =$ constant on $U$. $\square$

(Rmk: This also implies max principle for $Re(f)$, since $|e^f| = e^{Re(f)}$ has no (local) max.).

- One nice (non-local) consequence is a contraction principle: the Schwarz lemma.

Thm: $f$ analytic on $D = \{|z|<1\}$, and $|f(z)|<1$ $\forall z \in D$ (ie. $f: D \to D$), and $f(0)=0$, then $|f'(0)| \leq 1$, and $|f(z)| \leq |z|$ $\forall z \in D - \{0\}$.
Moreover if equality holds in either of these then $f(z) = e^{i\theta} z$ for some $e^{i\theta} \in S^1$.

Pf: Write $f(z) = \sum_{n=1}^{\infty} a_n z^n = z F(z)$ where $F(z) = \sum_{n=0}^{\infty} a_{n+1} z^n$ analytic
($f(0)=0 \Rightarrow$ no constant term)

For $|z| = r \in (0,1)$, we have $|F(z)| = \left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$, hence by the maximum principle, $|F(z)| \leq \frac{1}{r}$ whenever $|z| \leq r$. Taking $r \to 1$, $|F(z)| \leq 1$ $\forall z \in D$. Hence the bounds on $f'(0) = F(0)$ and $f(z) = zF(z)$. Moreover, if $|F|=1$ is achieved anywhere inside $D$ then $F$ is constant $= e^{i\theta}$, so $f(z) = e^{i\theta} z$. $\square$

Note:
- The bound on $|f'(0)|$ is the same as the bound one gets from Cauchy's integral formula. The Schwarz lemma is a strengthening to pointwise bounds $|f(z)| \leq |z|$ globally on the disc.
- by composing $f$ with fractional linear transformations, we can get Schwarz-type bounds for all sorts of other situations, eg. if $f$ maps a disc to a half-plane, etc.

- In fact, we have a stronger local result, the open mapping principle ($\Rightarrow$ max. principle).

**Thm:** A nonconstant analytic function is an open mapping, ie. $U$ open $\Rightarrow f(U)$ open

in other terms: $f$ analytic at $z_0$ $\Rightarrow \forall r>0, \exists \varepsilon>0$ st. $f(B_r(z_0)) \supset B_\varepsilon(f(z_0))$
non-constant
($\Rightarrow |f(z)|$, $\mathrm{Re}\, f(z)$, ... can't have local max)

First we prove

**Prop:** if $f(z)$ has an isolated zero at $z=z_0$, then $\exists$ analytic function $g$ defined near $z_0$, with $g(z_0)=0$, $g'(z_0)\neq 0$, and $n\geq 1$, st. $f(z)=g(z)^n$.

**Pf:** let $n=$ order of the zero of $f$, ie. write $f(z)=\sum_{k=n}^{\infty} a_k (z-z_0)^k = a_n (z-z_0)^n (1+h(z))$ with $h(z_0)=0$. $\exists V \ni z_0$ st. $|h(z)|<1$ $\forall z\in V$; over $V$ we can define
$g(z)=a_n^{1/n}(z-z_0)(1+h(z))^{1/n}$, where $(1+h(z))^{1/n}=\exp\left(\frac{1}{n}\log(1+h(z))\right)$ well def.d for $|h|<1$. $\square$

**Pf. thm:** for $z_0\in U$, write $f(z)-f(z_0)=g(z)^n$ for some $n\geq 1$, $g(z_0)=0$, $g'(z_0)\neq 0$.
By inverse function thm, $g$ is a local diffeo at $z_0$ (since $g'(z_0)\neq 0$), hence an open mapping near $z_0$ ($\exists$ continuous, actually analytic, inverse mapping), so $\forall V\ni z_0$ open ($\subset$ domain of $g$), $g(V)\ni 0$ contains some ball $B_\varepsilon(0)$, hence taking $n^{th}$ power, $f(V)\supset B(f(z_0),\varepsilon^n)$. $\square$