Last time we've seen Cauchy's formula ⇒ mean value identity, implying the maximum principle:

Thm: If f is analytic on $U \supset \overline{B_r(z)}$ then $f(z)$ is the average value of f on $S^1(z,r)$.

Thm: If f is analytic on $U \subset \mathbb{C}$ (open connected) & nonconstant, then $|f|$ doesn't achieve its maximum anywhere in U. In particular, if f is analytic on U and continuous on $\bar{U}$, $\bar{U}$ compact, then the maximum of $|f|$ on $\bar{U}$ is achieved on the boundary of U.

(Rmk: This also implies max principle for $\mathrm{Re}(f)$, since $|e^f| = e^{\mathrm{Re}(f)}$ has no (local) max.).

- One nice (non-local) consequence is a contraction principle: the Schwarz Lemma.

Thm: f analytic on $D = \{|z|<1\}$, and $|f(z)|<1$ $\forall z\in D$ (ie. $f: D\to D$), and $f(0)=0$, then $|f'(0)|\leq 1$, and $|f(z)|\leq |z|$ $\forall z\in D-\{0\}$.
Moreover if equality holds in either of these then $f(z) = e^{i\theta}z$ for some $e^{i\theta}\in S^1$.

Pf: Write $f(z) = \sum_{n=1}^{\infty} a_n z^n = z F(z)$ where $F(z) = \sum_{n=0}^{\infty} a_{n+1} z^n$ analytic
($f(0)=0 \Rightarrow$ no constant term)

For $|z| = r\in(0,1)$, we have $|F(z)| = \left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$, hence by the maximum principle, $|F(z)|\leq \frac{1}{r}$ whenever $|z|\leq r$. Taking $r\to 1$, $|F(z)|\leq 1$ $\forall z\in D$.
Hence the bounds on $f'(0) = F(0)$ and $f(z) = zF(z)$. Moreover, if $|F|=1$ is achieved anywhere inside D then F is constant $= e^{i\theta}$, so $f(z) = e^{i\theta}z$. □

Note:
- the bound on $|f'(0)|$ is the same as the bound one gets from Cauchy's integral formula. The Schwarz lemma is a strengthening to pointwise bounds $|f(z)|\leq|z|$ globally on the disc.
- by composing f with fractional linear transformations, we can get Schwarz-type bounds for all sorts of other situations, eg. if f maps a disc to a half-plane, etc.

There is another important class of functions which satisfy mean value & max. principle:

Def: A C^2 function $f: U\subset\mathbb{R}^n \to \mathbb{R}$ is harmonic if $\underbrace{\Delta}_{\text{Laplacian}} f = \sum \frac{\partial^2 f}{\partial x_i^2} = 0$.

(Physically important! eg. electric & gravitational potentials in vacuum are harmonic, so is temperature distribution at thermal equilibrium; etc.).

Real analysis gives general methods for studying harmonic functions, but the case of 2 real variables $f(x,y)$ is closely related to complex analysis.

- $u: U\subset\mathbb{C} \xrightarrow[C^2]{} \mathbb{R}$ is harmonic if $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial z\partial\bar{z}} = 0$

$\left[\text{using: } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \text{ so } \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$

$\Rightarrow$ Thm: If $f = u + iv$ is analytic, then $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are harmonic.

Pf. 1: Cauchy-Riemann eq^n $\frac{\partial f}{\partial \bar z} = 0$ says $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, so

$$\Delta u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = 0.$$

Pf. 2: $u = \frac{1}{2}(f + \bar f)$, $\Delta f = 4\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar z}\right) = 0$, $\Delta(\bar f) = 4\frac{\partial}{\partial \bar z}\left(\frac{\partial \bar f}{\partial z}\right) = 0$.

What is unique about harmonic functions in 2 variables is that we have a converse:

Thm: If u is harmonic on a simply-connected open $U \subset \mathbb{C}$, then there exists an analytic function $f: U \to \mathbb{C}$ s.t. $u = \operatorname{Re}(f)$.

ie. there exists a harmonic $v: U \to \mathbb{R}$ ("harmonic conjugate of u") s.t. $u + iv$ is analytic.

Ex: $u = \log|z| = \operatorname{Re}(\log z)$ on domain not enclosing origin $\rightsquigarrow v = \arg(z)$.

This shows the assumption on U is necessary (v not single-valued on $\mathbb{C}^*$).

(won't prove in lecture)

Pf: Given u harmonic, let $\alpha = 2\frac{\partial u}{\partial z} dz = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(dx + i\,dy)$ complex-valued 1-form.

Then α is closed, since $d\alpha = 2\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial z}\right) d\bar z \wedge dz$ and $2\frac{\partial^2 u}{\partial \bar z \partial z} = \frac{1}{2}\Delta u = 0$.

Or, in more familiar terms using real differentials:

$\operatorname{Re}(\alpha) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = du$ is exact hence closed

$\operatorname{Im}(\alpha) = \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dy$ is closed using $\Delta u = 0$.

Since U is simply connected, closed 1-forms on U are exact:

$\exists f: U \to \mathbb{C}$ st. $df = \alpha$. (or equivalently, $\exists v$ st. $dv = \operatorname{Im}(\alpha)$ and then $d(u + iv) = \alpha$).

(Construct f by integration: $f(z) = \int_{z_0}^{z} \alpha$, path-independent by Stokes since $d\alpha = 0$ and U simply conn^d)

Now $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z = 2\frac{\partial u}{\partial z} dz$ $\Rightarrow$ $\frac{\partial f}{\partial \bar z} = 0$ ie. f is analytic,

no $d\bar z$ term

and $d(\operatorname{Re} f) = \operatorname{Re}(\alpha) = du$ so up to adding a constant we can ensure $\operatorname{Re}(f) = u$. □

Now we know that harmonic functions are secretly real parts of analytic functions, we get:

Corollaries: (also true in n variables?)

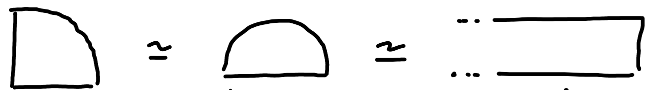
- any C^2 harmonic function is actually C^∞
- harmonic functions satisfy the mean value thm: $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta$.
- —— " —— maximum principle

Another pair of deep results (which we won't prove) are about existence of analytic mappings & harmonic functions:

* The Riemann mapping theorem:

Thm: if $U \subset \mathbb{C}$ is a non-empty simply connected open subset, $U \neq \mathbb{C}$, then there exists a biholomorphism $\varphi : U \xrightarrow{\sim} D = \{|z|<1\}$ ie. analytic bijection w/ analytic inverse.

Ex: can you find explicit bihol'm's



quarter disc $\simeq$ half disc $\simeq$ $(-\infty,0)\times(0,1)$?

or... unit disc $\simeq$ half-plane $\simeq$ $\mathbb{R}\times(0,1)$? (see HW).

* The existence of solutions to Dirichlet's problem (harmonic functions with prescribed values at the boundary of a domain) can be thought of as an analogue for harmonic functions:

Thm: if $U \subset \mathbb{C}$ is a simply connected bounded open subset with suff. nice boundary (eg. ∂U piecewise smooth) and $f \in C^0(\partial U, \mathbb{R})$ any continuous function
$\Rightarrow \exists$ unique function $u \in C^0(\bar{U}, \mathbb{R})$ st. $\begin{cases} u_{|\partial U} = f \\ u \text{ is harmonic inside } U. \end{cases}$

(uniqueness follows easily from the max-principle: $u - v = 0$ at ∂U, $u-v$ harmonic $\Rightarrow u - v \equiv 0$).

(One way to prove this is actually to first establish it for the unit disc, using Fourier series to reduce to trigonometric polynomials; $\sum c_n e^{in\theta} \rightsquigarrow \sum_{n\geq 0} c_n z^n + \sum_{n<0} c_n \bar{z}^{|n|}$ then use Riemann mapping theorem (+ slightly more) to map $U \xrightarrow[\sim]{\varphi} D$; u is harmonic iff $u \circ \varphi$ is.)

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* Back to analytic fns, there's a stronger local result: the open mapping principle ($\Rightarrow$ max. principle).

Thm: A nonconstant analytic function is an open mapping, ie. U open $\Rightarrow f(U)$ open

in other terms: f analytic at z_0 (non-constant) $\Rightarrow \forall r>0, \exists \varepsilon>0$ st. $f(B_r(z_0)) \supset B_\varepsilon(f(z_0))$
($\Rightarrow |f(z)|, \operatorname{Re} f(z), \ldots$ can't have local max)

First we prove

Prop: if $f(z)$ has an isolated zero at $z = z_0$, then $\exists$ analytic function g defined near z_0, with $g(z_0)=0$, $g'(z_0)\neq 0$, and $n \geq 1$, st. $f(z) = g(z)^n$.

Pf: let $n :=$ order of the zero of f, ie. write $f(z) = \sum_{k=n}^{\infty} a_k (z-z_0)^k = a_n (z-z_0)^n (1+h(z))$ with $h(z_0)=0$. $\exists V \ni z_0$ st. $|h(z)|<1 \ \forall z \in V$; over V we can define
$g(z) = a_n^{1/n} (z-z_0)(1+h(z))^{1/n}$, where $(1+h(z))^{1/n} = \exp\left(\frac{1}{n}\log(1+h(z))\right)$ well def'd for $|h|<1$. □

Pf. thm: for $z_0 \in U$, write $f(z) - f(z_0) = g(z)^n$ for some $n \geq 1$, $g(z_0)=0$, $g'(z_0)\neq 0$.
By inverse function thm, g is a local diffeo at z_0 (since $g'(z_0)\neq 0$), hence an open mapping near z_0 ($\exists$ continuous, actually analytic, inverse mapping), so $\forall V \ni z_0$ open ($\subset$ domain of g), $g(V) \ni 0$ contains some ball $B_\varepsilon(0)$, hence taking n^{th} power, $f(V) \supset B(f(z_0), \varepsilon^n)$. □

The argument principle: Our proof of open mapping principle actually shows: near z_0, f takes every value near $f(z_0)$ n times where n is the order of the zero of $f(z)-f(z_0)$ at $z=z_0$.

More generally, we can estimate the number of zeros of f (or $\#f^{-1}(c)$) in a domain D:

Thm: If $f: U \to \mathbb{C}$ is analytic, D bounded domain with $\bar{D} \subset U$, $\partial D = \gamma$ piecewise smooth, assume f is nonzero at every point of γ. Then the number of zeros of f inside D, counted with multiplicity = order of each zero, is $n(\gamma,0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$.

Observe: $\frac{f'(z)}{f(z)} = \frac{d}{dz}(\log f(z))$ - the <u>logarithmic derivative</u>.

(NB: $\log f$ is only def^{d} locally up to $+2\pi i\mathbb{Z}$, but this doesn't matter for the derivative!).

Let $z_1, \dots, z_k$ be the zeros of f inside D, with multiplicities $m_1, \dots, m_k$.
(isolated, hence finitely many since $\bar{D}$ is compact).

Then we can write $f(z) = (z-z_1)^{m_1} \dots (z-z_k)^{m_k}\, g(z)$ where g is analytic and nowhere zero in D (check this makes sense & works near each z_i).

Properties of log (or calculation) $\Rightarrow$ $\frac{f'(z)}{f(z)} = \frac{m_1}{z-z_1} + \dots + \frac{m_k}{z-z_k} + \frac{g'(z)}{g(z)}$.

Now $\frac{g'(z)}{g(z)}$ is analytic in D (g has no zeroes) so $\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$,

while $\frac{1}{2\pi i}\int_\gamma \frac{m_j}{z-z_j}\,dz = m_j$ (Cauchy formula) $\Rightarrow$ $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum m_j$. $\square$

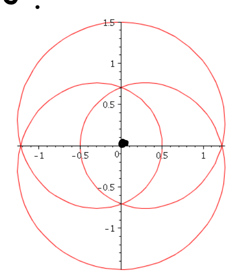
* <u>Topological / geometric interpretation</u>:
view f as a mapping $U \to \mathbb{C}$, it maps the loop $\gamma \subset U$ to $f_*(\gamma) = f\circ\gamma$ loop in $\mathbb{C}$. (may self-intersect). We've assumed $f \neq 0$ on γ, so $f\circ\gamma$ is actually a loop in $\mathbb{C}^*$.

$$n(\gamma,0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} \underset{\text{set } w=f(z)}{=} \frac{1}{2\pi i}\int_{f_*(\gamma)} \frac{dw}{w}$$
(pullback formula, or more concretely, change of var's in path integral / chain rule)

$= $ change in $\frac{1}{2\pi i}\log(w)$, ie. $\frac{1}{2\pi}\arg(w)$ around $f_*(\gamma)$

$= $ <u>winding number</u> of $f\circ\gamma$ around the origin in 0.

<u>Ex</u>: $f(z) = z^3 - \frac{1}{2}z$ on unit circle: winding number around origin is 3 (3 roots in unit disc)



<u>Generalization</u>: if $c \notin f(\gamma)$ then $n(\gamma,c) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-c}\,dz$ = winding number of $f(\gamma)$ around $c \in \mathbb{C}$ gives the number of times $f(z)=c$ inside D (with multiplicities).

This quantity varies continuously with c, & is an integer $\Rightarrow$ locally <u>constant</u> (indep^{t} of c) as long as $c \notin f(\gamma)$. (Note: γ is compact, so $f(\gamma)$ as well $\Rightarrow$ $\mathbb{C} - f(\gamma)$ is <u>open</u>.

Applying to $\gamma = S^1(z,\delta)$, $n(\gamma, f(z)) > 0$ (isolation of zeros $\Rightarrow$ for $\delta > 0$ small, $f(z) \notin f(\gamma)$.) ⑤

$\Rightarrow n(\gamma, w) > 0 \quad \forall w \in B_\varepsilon(f(z)) \subset \underset{\text{open}}{\mathbb{C} \setminus f(\gamma)}$, i.e. $f(B_\delta(z)) \supset B_\varepsilon(f(z))$. (in fact the whole connected component of $f(z)$ in $\mathbb{C} \setminus f(\gamma)$).

This gives another proof of the <u>open mapping principle</u>.