Math 55b Homework 11

Due Wednesday April 21, 2021.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.

Material covered: Maximum principle, harmonic functions; argument principle, residues and integration (Ahlfors sections 4.3, 4.5 and 4.6.1, or McMullen sections 11 and 13)

1. Let f be an analytic function over a domain U which contains the closed disc $\{z \in \mathbb{C},\ |z| \leq 3\}$, and suppose that $f(1) = f(i) = f(-1) = f(-i) = 0$. Show that $|f(0)| \leq \dfrac{1}{80} \max\limits_{|z|=3} |f(z)|$, and find all functions for which equality holds.

2. Let $f(z)$ be an analytic function over the annulus $R_1 < |z| < R_2$, and let $M(r) = \sup_{z \in S^1(r)} |f(z)|$. Prove that $\log M(e^s)$ is a convex function of $s \in (\log R_1, \log R_2)$.

(Hint: apply the maximum principle to $z^m f(z)^n$ for suitably chosen m, n.)

3. (a) We consider the following three domains in $\mathbb{C}$: $D = \{|z| < 1\}$, $H = \{\mathrm{Re}(z) > 0\}$, and $S = \{0 < \mathrm{Im}(z) < 1\}$ (the unit disc, the right half-plane, and an infinite horizontal strip), and their closures in $\mathbb{C}$. Find explicit homeomorphisms $\overline{D} - \{\pm 1\} \simeq \overline{H} - \{0\} \simeq \overline{S}$ whose restrictions to the interior are biholomorphisms (i.e. analytic maps with analytic inverses) $D \simeq H \simeq S$.

(b) Show that there exists a unique continuous function $u : \overline{D} - \{\pm 1\} \to \mathbb{R}$ such that u is harmonic in D, $u(z) = 1$ on the upper half of the unit circle ($|z| = 1$ and $\mathrm{Im}(z) > 0$), and $u(z) = 0$ on the lower half of the unit circle ($|z| = 1$ and $\mathrm{Im}(z) < 0$).

4. Determine explicitly the largest disk centered at the origin on which the mapping $w = z^2 + z$ is injective (one-to-one).

5. How many roots of the equation $z^7 + 7z^4 - 3z^2 + 2 = 0$ satisfy $1 < |z| < 2$?

(Hint: use Rouché's theorem).

6. Find the Laurent series expressions for the function $f(z) = \dfrac{1}{z(z-1)(z-2)}$:

(i) in the region $\{1 < |z| < 2\}$, (ii) in the region $\{|z| > 2\}$.

7. Find all the singularities of $f(z) = z/(e^{z^2} - 1)$, and determine the residues at each of its poles.

8. Evaluate the following integrals by the method of residues:

$$\text{(a)} \int_0^{\pi/2} \frac{dx}{a + \sin^2 x} \ (a > 1), \qquad \text{(b)} \int_0^{\infty} \frac{x^2\, dx}{x^4 + 5x^2 + 6}, \qquad \text{(c)} \int_0^{\infty} \frac{\cos x}{x^2 + a^2}\, dx \ (a > 0)$$

9. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?