The argument principle = formula for the number of zeros of f (or $\#f^{-1}(c)$) in a domain D:

Thm: If $f: U \to \mathbb{C}$ is analytic, D bounded domain with $\bar{D} \subset U$, $\partial D = \gamma$ piecewise smooth, assume f is nonzero at every point of γ. Then the number of zeros of f inside D, counted with multiplicity = order of each zero, is $n(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz$.

Observe: $\frac{f'(z)}{f(z)} = \frac{d}{dz}(\log f(z))$ – the logarithmic derivative.
(NB: $\log f$ is only def^{d} locally up to $+2\pi i\mathbb{Z}$, but this doesn't matter for the derivative!).

Let $z_1, \dots, z_k$ be the zeros of f inside D, with multiplicities $m_1, \dots, m_k$.
(isolated, hence finitely many since $\bar{D}$ is compact).

Then we can write $f(z) = (z-z_1)^{m_1} \dots (z-z_k)^{m_k}\, g(z)$ where g is analytic and nowhere zero in D (check this makes sense & works near each z_i).

Properties of log (or calculation) $\Rightarrow$ $\frac{f'(z)}{f(z)} = \frac{m_1}{z-z_1} + \dots + \frac{m_k}{z-z_k} + \frac{g'(z)}{g(z)}$.

Now $\frac{g'(z)}{g(z)}$ is analytic in D (g has no zeroes) so $\int_\gamma \frac{g'(z)}{g(z)}\, dz = 0$,
while $\frac{1}{2\pi i}\int_\gamma \frac{m_j}{z-z_j}\, dz = m_j$ (Cauchy formula) $\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum m_j$. $\square$

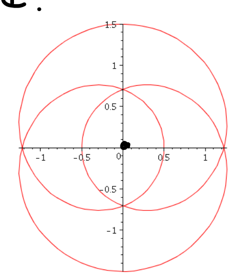
* Topological / geometric interpretation:
view f as a mapping $U \to \mathbb{C}$, it maps the loop $\gamma \subset U$ to $f_*(\gamma) = f\circ\gamma$ loop in $\mathbb{C}$.
(may self-intersect). We've assumed $f \neq 0$ on γ, so $f\circ\gamma$ is actually a loop in $\mathbb{C}^*$.

$$n(\gamma,0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} \underset{\text{set } w=f(z)}{=} \frac{1}{2\pi i}\int_{f_*(\gamma)} \frac{dw}{w}$$ (pullback formula, or more concretely, change of var's in path integral / chain rule)

$= $ change in $\frac{1}{2\pi i}\log(w)$, ie. $\frac{1}{2\pi}\arg(w)$ around $f_*(\gamma)$

$= $ winding number of $f\circ\gamma$ around the origin in $\mathbb{C}$.

Ex: $f(z) = z^3 - \frac{1}{2}z$ on unit circle:
winding number around origin is 3 (3 roots in unit disc)



Generalization: if $c \notin f(\gamma)$ then $n(\gamma, c) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-c}\, dz$ = winding number of $f(\gamma)$ around $c \in \mathbb{C}$ gives the number of times $f(z) = c$ inside D (with multiplicities).

This quantity varies continuously with c, & is an integer $\Rightarrow$ locally constant (indep^{t} of c) as long as $c \notin f(\gamma)$. (Note: γ is compact, so $f(\gamma)$ as well $\Rightarrow \mathbb{C} - f(\gamma)$ is open.

Applying to $\gamma = S^1(z,\delta)$, $n(\gamma, f(z)) > 0$ (isolation of zeros $\Rightarrow$ for $\delta>0$ small, $f(z) \notin f(\gamma)$) ②

$\Rightarrow n(\gamma, w) > 0 \quad \forall w \in B_\varepsilon(f(z)) \subset \underset{\text{open}}{\mathbb{C} - f(\gamma)}$, i.e. $f(B_\delta(z)) \supset B_\varepsilon(f(z))$. (in fact the whole connected component of $f(z)$ in $\mathbb{C} - f(\gamma)$).

This gives another proof of the <u>open mapping principle</u>.

* Another immediate generalization is to the case where f is <u>meromorphic</u> in D, rather than analytic: similarly write
$$f(z) = \frac{(z-a_1)^{m_1} \cdots (z-a_k)^{m_k}}{(z-b_1)^{n_1} \cdots (z-b_\ell)^{n_\ell}}\, g(z),$$
where
a_j are the zeros of f in D (with order m_j)
b_j —"— poles ——"—— n_j
$$\Rightarrow \text{winding}(f\circ\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} = \sum_{j=1}^{k} m_j - \sum_{j=1}^{\ell} n_j.$$

* A useful consequence of the argument principle is

<u>Rouché's thm:</u> if f and g are analytic in $U \supset \overline{D}$, $\partial D = \gamma$ simple closed curve, and $|f(z) - g(z)| < |f(z)|$ $\forall z \in \gamma$, then f and g have the same number of zeroes in D, counting with multiplicities.

<u>Proof:</u> $\left|\dfrac{g(z)}{f(z)} - 1\right| < 1$ on γ, so $\frac{g}{f}$ maps γ to the open disc $B_1(1)$, which doesn't enclose the origin. So the winding number $=$ #zeros $-$ #poles $= \#g^{-1}(0) - \#f^{-1}(0) = 0$. □

Rouché's thm is a good way of estimating the number of zeros of g in D by reducing to an easier calculation.

<u>Ex:</u> $g(z) = z^3 - 4z^2 + 1$: the fundamental thm of algebra says g has 3 roots, but how many of these are in the unit disc?
Answer: on S^1, $|z^3+1| < |4z^2|$, so we can compare to $f(z) = -4z^2$ and conclude 2 of the 3 roots are in the unit disc.

<u>Residue calculus:</u> instead of using Cauchy's integral formula to study the behavior of analytic functions, let's now use it to evaluate integrals!

Assume we want to evaluate $\int_\gamma f(z)\,dz$, where $\gamma = \partial D$ and f is analytic in $U \supset \overline{D} - \{p_1, \dots, p_n\}$. (or, later, a definite integral whose value can be related to $\int_\gamma$).

* <u>Def:</u> The <u>residue</u> of f at p is $\operatorname{Res}_p(f) = \dfrac{1}{2\pi i}\displaystyle\int_{S^1(p,\varepsilon)} f(z)\,dz$.
(for $\varepsilon > 0$ small so f is analytic in $D^*(p,\varepsilon) = D(p,\varepsilon) - \{p\}$.

Expressing f as a Laurent series $\sum_{-\infty}^{\infty} a_n (z-p)^n$ in $D^*(p,\varepsilon)$, $\boxed{\operatorname{Res}_p(f) = a_{-1}}$

So: the residue is easiest to calculate if f has a simple pole (ie. order 1) at p, in this case $\operatorname{Res}_p(f) = \lim_{z\to p} (z-p)f(z)$. Otherwise, need to calculate, usually by determining part of the Laurent series for f. (eg. for rational functions, partial fraction decomposition will accomplish this).

* Now, Cauchy's theorem for $D \setminus \bigcup D(p,\varepsilon)$ gives:

<u>Residue theorem</u>: $\bar{D}$ compact domain with piecewise smooth boundary $\gamma = \partial D$, $P \subset \operatorname{int}(D)$ finite set, f analytic on $U \supset \bar{D} - P$, then $\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{p\in P} \operatorname{Res}_p(f)$.

We now explore how to use this to evaluate various kinds of definite integrals.

<u>Example 1</u>: $\int_0^{2\pi} R(\sin\theta, \cos\theta)\,d\theta$ (or $R(e^{i\theta})$) where R is a rational function (w/o. poles on S^1).

<u>e.g.</u> let's calculate $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$, where $a>1$.

Set $z = e^{i\theta}$ to turn this into a path integral on S^1: then $d\theta = \frac{1}{i}\, d\log z = \frac{dz}{iz}$ and $\cos\theta = \frac{z+z^{-1}}{2}$. $\Rightarrow \int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \int_{S^1} \frac{dz/z}{\frac{i}{2}(z+2a+z^{-1})} = -2i \int_{S^1} \frac{dz}{z^2+2az+1}$

The poles are at $p_\pm = -a \pm \sqrt{a^2-1}$; of these, only $p_+ = -a+\sqrt{a^2-1}$ is inside the unit circle. How do we calculate the residue?

→ partial fractions: $f(z) = \frac{1}{(z-p_+)(z-p_-)} = \frac{1}{p_+-p_-}\left(\frac{1}{z-p_+} - \frac{1}{z-p_-}\right)$, so $\operatorname{Res}_{p_+}(f) = \frac{1}{p_+-p_-} = \frac{1}{2\sqrt{a^2-1}}$

→ since this is a simple pole: $\operatorname{Res}_{p_+}(f) = \lim_{z\to p_+} (z-p_+)f(z) = \lim_{z\to p_+} \frac{(z-p_+)}{(z-p_+)(z-p_-)} = \frac{1}{p_+-p_-}$ = same.

Hence $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = -2i\int_{S^1} f(z)\,dz = 4\pi \operatorname{Res}_{p_+}(f) = \frac{2\pi}{\sqrt{a^2-1}}$.

<u>Example 2</u>: $\int_{-\infty}^{\infty} f(x)\,dx$ where f is a rational function $\frac{P(x)}{Q(x)}$

(assume Q has no real roots, and $\deg Q \geq \deg P + 2$, so the integral converges).

The trick here is to recall $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$; and complete the segment $[-R,R]$ to a closed curve in $\mathbb{C}$ by adding a semicircle of radius R in the upper half plane: $\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{\substack{\operatorname{Im} p>0 \\ \text{and } |p|<R}} \operatorname{Res}_p(f)$.

Now, since $f = \frac{P}{Q}$ with $\deg Q \geq \deg P + 2$, $|f(z)| \leq \frac{C}{|z|^2}$, so $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$.

Hence: taking $R \to \infty$, we get $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\mathrm{Im}(p)>0} \mathrm{Res}_p(f)$.

We can use $\lim_{z\to p}(z-p)f(z)$ to find $\mathrm{Res}_p(f)$ if all poles are simple, else partial fractions. (Of course, the method of partial fractions already allowed us to integrate f !)

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i\, \mathrm{Res}_i\left(\frac{1}{z^2+1}\right) = \pi$ (which we already knew using arctan)

using $\displaystyle\mathrm{Res}_{z=i}\left(\frac{1}{z^2+1}\right) = \lim_{z\to i} \frac{z-i}{z^2+1} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}$.

Example 3: mixed rational & exponential functions (now we get to something new!)

Assume $f(z) = \dfrac{P(z)}{Q(z)}$ is a rational function as before (no real poles, $\deg Q \geq \deg P + 2$).

Then we can use the same method as above to calculate $\displaystyle\int_{-\infty}^{\infty} f(z)\,e^{iz}\,dz$ by considering a large disc in the upper half plane.

The key point is that $|e^{iz}| = e^{-\mathrm{Im}(z)} \leq 1$ in the upper half plane, so the path-integral along the semicircle still goes $\to 0$.

(whereas if integrand has e^{-iz} we'd want to consider the lower half-plane instead)

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2}\,dz = 2\pi i\,\mathrm{Res}_{z=i}\left(\frac{e^{iz}}{1+z^2}\right) = 2\pi i \cdot e^{-1}\cdot \mathrm{Res}_{z=i}\left(\frac{1}{1+z^2}\right) = \frac{2\pi i}{2ie} = \frac{\pi}{e}$.

Taking real and imaginary parts:

$\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{1+x^2} = \frac{\pi}{e}$, $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x\,dx}{1+x^2} = 0$ (this was expected, by symmetry)

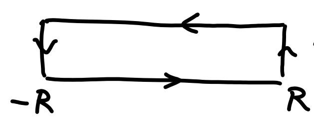
Example 3': we can actually handle the case $\deg Q = \deg P + 1$!! (still assuming $\nexists$ real poles)

Then $\displaystyle\int_{-\infty}^{\infty} f(z)\,e^{iz}\,dz$ still converges, but not absolutely!

(example: $\displaystyle\int_{n\pi}^{(n+1)\pi} \frac{x\sin x}{1+x^2}\,dx \sim (-1)^n\frac{2}{n}$ convergent series, even though not abs. convergent).

Closing the path in $\mathbb{C}$ also requires some care, to show the integrals along the portions we add do $\to 0$ as radius $\to \infty$: $\int_{\text{semicircle}} f(z)\,dz \not\to 0$ since $|f(z)| \sim \frac{C}{R}$ vs. length $= \pi R$.

One popular choice is to take a large rectangle



$\updownarrow$ $I \to \infty$ but $\ll R$.

but semicircle is actually fine! The point is that:

- over the portion where $\mathrm{Im}(z) > A$, $|e^{iz}| < e^{-A}$, so $\left|\int f(z)\,e^{iz}\,dz\right| \lesssim C e^{-A} \to 0$ as $A \to \infty$
- the portion where $\mathrm{Im}(z) < A$ has length $\lesssim A$, and $|z| \gtrsim R$, so we have a bound by $\dfrac{CA}{R}$.

If we use eg. $A = \sqrt{R}$ to split things, we still get $\to 0$ as $R \to \infty$.