Math 55b Homework 10

Due Wednesday April 14, 2021.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.

Material covered: Cauchy's integral formula and applications; poles and singularities (Ahlfors sections 4.1-4.3 and 5.1, or McMullen sections 10 and 12)

1. For $n, m \in \mathbb{Z}$, calculate: (a) $\displaystyle\int_{|z|=1} z^{-n}\, e^z\, dz$, (b) $\displaystyle\int_{|z|=2} z^n (1-z)^m\, dz$.

2. Calculate $\displaystyle\int_{|z|=r} \frac{|dz|}{|z-a|^2}$ for $|a| \neq r$. Here $|dz|$ stands for the length element, i.e. given a parametrized path $\gamma : [a, b] \to \mathbb{C}$ we define $\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))\, |d\gamma/dt|\, dt$.

(Hint: observe that, on the circle of radius r, $\bar{z} = r^2/z$ and $|dz| = -ir\, dz/z$.)

3. Show that if f is analytic in a neighborhood of 0 and n is a positive integer then there exists an analytic function g in a neighborhood of 0 such that $f(z^n) = f(0) + g(z)^n$.

4. Let $f_n(z)$ be a sequence of analytic functions on the unit disc $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$, converging uniformly to $f(z)$.

(a) Show that $f_n'(z)$ converges uniformly to $f'(z)$ on $\overline{D}_r = \{z \in \mathbb{C} \,|\, |z| \leq r\}$ for all $r < 1$.

(b) Give an example showing that $f_n'(z)$ does not necessarily converge uniformly to $f'(z)$ on D.

5. (a) Prove that, if $f(z)$ is analytic in the whole complex plane and there exist constants $a, R > 0$ and an integer $n \geq 0$ such that $|f(z)| \leq a|z|^n$ for all $z \in \mathbb{C}$ with $|z| \geq R$, then f is a polynomial.

(b) Prove that, if $f(z)$ is analytic in the whole complex plane and there exist constants $a, R > 0$ and an integer $n \geq 0$ such that $|f(z)| \geq a|z|^{-n}$ for all $z \in \mathbb{C}$ with $|z| \geq R$, then f is a polynomial.

(Hint for (a): consider Cauchy's bound for $f^{(n+1)}$. For (b): consider the singularity of $z^{-n} f(1/z)$ at the origin).

6. (a) Show that, if $f(z)$ is analytic in the unit disc $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ and $|f(z)| < 1$ for all $z \in D$, then

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \quad \text{for all } z \in D.$$

(Hint: use a fractional linear transformation to reduce to the case where $z = f(z) = 0$).

(b) Show that, if $f(z)$ is analytic in the disc $D_r = \{z \,|\, |z| < r\}$ and $\operatorname{Re} f(z) < a$ for all $z \in D_r$ for some constant $a \in \mathbb{R}$, then

$$|f'(z)| \leq \frac{2r(a - \operatorname{Re} f(z))}{r^2 - |z|^2} \quad \text{for all } z \in D_r.$$

(Hint: consider $g(z) = f(rz)/(2a - f(rz))$.)

7. Show that, if $f(z)$ is analytic in the whole complex plane and there exist constants $a, b > 0$ and an integer $n \geq 0$ such that $|\operatorname{Re} f(z)| \leq a|z|^n + b$ for all $z \in \mathbb{C}$, then f is a polynomial.

(Hint: Use the result of 6(b) to find a bound on $|f'(z)|$, then use the result of 5(a)).

8. Find all analytic functions $f(z)$ on the whole complex plane such that f never takes the value zero and there exist $c_1, c_2 > 0$ such that $c_1^{-1} \exp(-c_2|z|^2) \leq |f(z)| \leq c_1 \exp(c_2|z|^2)$ for all $z \in \mathbb{C}$.

9. Let $\sum a_n z^n$ be the Laurent series for $f(z) = 1/(e^z - 1)$ near $z = 0$. Find a_n for $n \leq 3$. What is the radius of convergence of this series?

10. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?