Recall: the residue of f (analytic in $\overline{D^*(p,\varepsilon)}$) at p is $\mathrm{Res}_p(f) = \frac{1}{2\pi i}\int_{S^1(p,\varepsilon)} f(z)\,dz$.
$=$ coeff^t of $(z-p)^{-1}$ in Laurent series of f near p ($= \lim (z-p)f(z)$ if simple pole)

Residue theorem: $\overline{D}$ compact domain with piecewise smooth boundary $\gamma = \partial D$, $P \subset \mathrm{int}(D)$ finite set, f analytic on $U \supset \overline{D} - P$, then $\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{p\in P} \mathrm{Res}_p(f)$.

We've used this to evaluate definite integrals, by completing path to a closed curve.

Example 3: mixed rational & exponential functions

Assume $f(z) = \frac{P(z)}{Q(z)}$ is a rational function without real poles, $\deg Q \geq \deg P + 2$.

Then we can calculate $\int_{-\infty}^{\infty} f(z)\,e^{iz}\,dz$ by considering a large semidisc in the upper half plane.

$$\int_{\partial D} f(z)\,e^{iz}\,dz = \int_{-R}^{R} + \int_{\text{semicircle}} = 2\pi i \sum_{|p|<R,\ \mathrm{Im}\,p>0} \mathrm{Res}_p\left(f(z)\,e^{iz}\right). \quad + \text{take limit as } R\to\infty.$$

The key point is that $|e^{iz}| = e^{-\mathrm{Im}(z)} \leq 1$ in the upper half plane, so the path-integral along the semicircle goes $\to 0$. ($|f(z)e^{iz}| < C/R^2$, length $= \pi R$).
(whereas if integrand has e^{-iz} we'd want to consider the lower half-plane instead.)

Ex: $$\int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2}\,dz = 2\pi i\,\mathrm{Res}_{z=i}\left(\frac{e^{iz}}{1+z^2}\right) = 2\pi i \cdot \left(\frac{e^{iz}}{z+i}\right)_{|z=i} = \frac{2\pi i\, e^{-1}}{2i} = \frac{\pi}{e}.$$

Taking real and imaginary parts:
$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{1+x^2} = \frac{\pi}{e}, \quad \int_{-\infty}^{\infty} \frac{\sin x\,dx}{1+x^2} = 0 \quad \text{(clear by symmetry)}$$

Example 3': we can actually handle the case $\deg Q = \deg P + 1$ (still assuming $\nexists$ real poles)

Then $\int_{-\infty}^{\infty} f(z)\,e^{iz}\,dz$ still converges, but not absolutely!

(example: $\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx \sim (-1)^n \frac{2}{n}$ convergent series, even though not abs. convergent).

Closing the path in $\mathbb{C}$ also requires some care, to show the integrals along the portions we add do $\to 0$ as radius $\to\infty$: bounding the integrand by $C/|z|$ isn't good enough.
One popular choice = large rectangle, but semicircle is actually fine! The point is that:

- over the portion where $\mathrm{Im}(z) > A$, $|e^{iz}| < e^{-A}$, so $\left|\int f(z)\,e^{iz}\,dz\right| \lesssim C e^{-A} \to 0$ as $A\to\infty$
- the portion where $\mathrm{Im}(z) < A$ has length $\lesssim A$, and $|z| \gtrsim R$, so we have a bound by $\frac{CA}{R}$.

If we use eg. $A = \sqrt{R}$ to split things, we still get $\to 0$ as $R\to\infty$.

Eg: $\displaystyle\int_{-\infty}^{\infty} \frac{ze^{iz}}{a^2+z^2}\,dz = 2\pi i \operatorname{Res}_{z=ia}\left(\frac{ze^{iz}}{a^2+z^2}\right) = 2\pi i \left(\frac{ze^{iz}}{z+ia}\right)_{|z=ia} = i\pi e^{-a}.$
$(a>0)$

Taking imaginary part, $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{a^2+x^2}\,dx = \pi e^{-a}.$

How about... $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$? (G.H. Hardy's note in the Mathematical Gazette, 1909, scores various methods (!)).

→ this one again converges, though not absolutely.

→ $\displaystyle\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$ is analytic in the whole plane, so... what residues ??

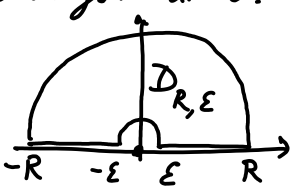
→ $\displaystyle\frac{\sin z}{z} = \frac{e^{iz}-e^{-iz}}{2iz} \to \infty$ both in upper & lower half plane, so how do we use our trick of closing to a half-disc?

→ however... $\displaystyle\frac{\sin x}{x} = \lim_{a\to 0} \frac{x\sin x}{a^2+x^2}$, and in fact after a painful discussion of the convergence as $a\to 0$ and interchange of limits, one can check $a\to 0$ is legitimate.

But it is more instructive to see how we can adjust the previous argument to handle $a=0$.

→ the actual issue: for $x\in\mathbb{R}$, $\displaystyle\frac{\sin x}{x} = \operatorname{Im}\left(\frac{e^{ix}}{x}\right)$, but $\displaystyle\frac{e^{iz}}{z}$ has a pole at 0, on the path of integration. And in fact, $\displaystyle\int_0^{\infty} \frac{e^{ix}}{x}\,dx$ is divergent at 0.

Solution: modify the contour of integration to avoid 0, to carve out a small half-disc from our large semidisc or rectangle.



Observe:

- $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{\sin x}{x}\,dx = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \operatorname{Im} \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{e^{iz}}{z}\,dz$

- $\displaystyle\int_{\partial D_{R,\varepsilon}} \frac{e^{iz}}{z}\,dz = 0$ by Cauchy (no poles in $D_{R,\varepsilon}$)

- the integral on the semicircle of radius R tends to 0 as $R\to\infty$ as before ($\displaystyle\left|\frac{e^{iz}}{z}\right| = \frac{e^{-\operatorname{Im} z}}{R}$ ⇒ consider separately regions $\operatorname{Im}(z)<A$ and $>A$ for $1 \ll A \ll R$.)

- on the semicircle of radius ε: $\operatorname{Res}_0\left(\frac{e^{iz}}{z}\right) = 1$, so we can write $\displaystyle\frac{e^{iz}}{z} = \frac{1}{z} + g(z)$ where $g(z)$ is analytic near 0 $\left(g(z) = \frac{e^{iz}-1}{z}\right)$.

Since g is bounded, $\displaystyle\int_{C_\varepsilon} g(z)\,dz \to 0$ as $\varepsilon\to 0$, whereas $\displaystyle\int_{C_\varepsilon} \frac{1}{z}\,dz = i\pi$ (half of our usual $2\pi i$!).

Combining: $\partial D_{R,\varepsilon} = ([-R,-\varepsilon]\cup[\varepsilon,R]) + C_R - C_\varepsilon$

$\displaystyle\Rightarrow \lim_{\varepsilon\to 0,\, R\to\infty} \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{e^{iz}}{z}\,dz = i\pi \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.$

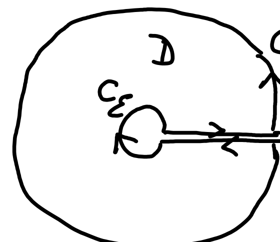
- One more class of examples: <u>non-integer powers of z</u>.

Consider for example: $\displaystyle\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx$ for $0<\alpha<1$. ($\Rightarrow$ converges at ∞).

If $\alpha = \frac{p}{q} \in \mathbb{Q}$ we can evaluate by substitution $x = u^q$ to get a rational function (but still not much fun). But here's a more general approach.

Difficulty: $\dfrac{z^\alpha}{1+z^2}$ isn't a single-valued analytic function $\Rightarrow$ how to use residues?

<u>Trick</u>: do something similar to our last example, use a "keyhole" region of integration: $\varepsilon \le |z| \le R$, with a slit along real positive axis.



With a better behaved integrand, the two portions along $[\varepsilon, R]$ would cancel out! But here they don't; we can define $\dfrac{z^\alpha}{1+z^2}$ as an analytic function over $\mathrm{int}(D)$, but its values on either side of the real axis don't match!

Explicitly: we take $z^\alpha = e^{\alpha \log z}$ to be the branch with $\mathrm{Im}(\log z) \in (0, 2\pi)$. Going around the origin, $\log x \rightsquigarrow \log x + 2\pi i$, so x^α gets multiplied by $e^{2\pi i \alpha}$.

So $$\int_{\partial D} \frac{z^\alpha}{1+z^2}\,dz = \int_\varepsilon^R \frac{x^\alpha}{1+x^2}\,dx + \int_{C_R} \frac{z^\alpha}{1+z^2}\,dz - \int_\varepsilon^R \frac{e^{2\pi i\alpha} x^\alpha}{1+x^2}\,dx - \int_{C_\varepsilon} \frac{z^\alpha}{1+z^2}\,dz$$

$\displaystyle\int_{C_R} \to 0$ as $R \to \infty$ $\left(\left|\frac{z^\alpha}{1+z^2}\right| \le \frac{C}{R^{2-\alpha}},\ \text{length} = 2\pi R, \text{ and } 2-\alpha > 1 \right)$.

$\displaystyle\int_{C_\varepsilon} \to 0$ as $\varepsilon \to 0$ (integrand and length $\to 0$).

So: $$\lim_{\varepsilon\to 0,\, R\to\infty} \int_{\partial D} \frac{z^\alpha}{1+z^2}\,dz = \left(1 - e^{2\pi i \alpha}\right) \int_0^\infty \frac{x^\alpha}{1+x^2}\,dx.$$

while the residue formula equates this with $2\pi i \left(\mathrm{Res}_{z=i} \left(\dfrac{z^\alpha}{1+z^2}\right) + \mathrm{Res}_{z=-i} \right)$

at $z=i$: $\displaystyle\lim_{z\to i} \frac{z^\alpha (z-i)}{z^2+1} = \left(\frac{z^\alpha}{z+i}\right)_{|z=i} = \frac{1}{2i} e^{\alpha \log(i)} = \frac{1}{2i} e^{i\frac{\pi}{2}\alpha}$

at $z=-i$: similarly get $-\dfrac{1}{2i} e^{3i\frac{\pi}{2}\alpha}$.

Hence: $$\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \pi \frac{e^{i\pi\alpha/2} - e^{3i\pi\alpha/2}}{1 - e^{2\pi i\alpha}} = \frac{\pi \sin\left(\frac{\pi\alpha}{2}\right)}{\sin(\pi\alpha)} = \frac{\pi}{2\cos(\pi\alpha/2)}.$$

Our next topic is infinite sum & product expansions (Ahlfors ch. 5.1–5.2)

We've seen: if f is analytic in the annulus $\{R_1 < |z| < R_2\}$ then it has a Laurent series expansion $f(z) = \sum_{-\infty}^{\infty} a_n z^n$, which may or may not have a finite negative part.

If the inner radius is $R_1 = 0$, then finite negative part $\iff$ pole at $z=0$
infinite —"— $\iff$ essential singularity

But if $R_1 > 0$ this need not be the case!

Ex: $\frac{1}{1-z}$ has a pole at $z=1$. So we have two different Laurent series:

- for $|z|<1$, $\frac{1}{1-z} = 1 + z + z^2 + \dots$ $(R_2 = 1)$
- for $|z|>1$, $\frac{1}{1-z} = \frac{-1}{z(1-\frac{1}{z})} = -z^{-1} - z^{-2} - z^{-3} - \dots$ $(R_1 = 1)$.

Of course, in this example (and for most rational functions) Laurent series aren't the best choice of representation. Partial fractions make more sense, or product expansions.

- Products: $R(z) = \frac{P(z)}{Q(z)} \Rightarrow$ can factor $R(z) = c \frac{\prod_{i=1}^{k} (z-a_i)^{n_i}}{\prod_{i=1}^{\ell} (z-b_i)^{m_i}}$
- Sums (partial fractions): if the poles are all simple, can write

$$R(z) = \frac{c_1}{z-b_1} + \dots + \frac{c_\ell}{z-b_\ell} + S(z) \quad \text{where } c_i \in \mathbb{C},\ S(z) \text{ polynomial.}$$

or in general, $R(z) = \frac{C_1(z)}{(z-b_1)^{m_1}} + \dots + \frac{C_\ell(z)}{(z-b_\ell)^{m_\ell}} + S(z)$

where $C_1, \dots, C_\ell, S$ are polynomials, $\deg(C_i) \le m_i - 1$.

We'll learn how to find similar (infinite) sum or product expansions for general meromorphic functions.

Starting point: if $f(z)$ is meromorphic with a pole of order m at $b \in \mathbb{C}$, then we can write $f(z) = \frac{g(z)}{(z-b)^m}$ with $g(z)$ analytic in a nbd. of b.

Or, expressing $g(z)$ as a power series in $(z-b)$, $g(z) = \sum_{n=0}^{\infty} a_n (z-b)^n$ we have a Laurent series for f with finite negative part, as already noticed:

$$f(z) = \left[\frac{a_0}{(z-b)^m} + \frac{a_1}{(z-b)^{m-1}} + \dots + \frac{a_{m-1}}{z-b}\right] + h(z), \quad h(z) = \sum_{n=0}^{\infty} a_{m+n}(z-b)^n$$

analytic near b.

THE POLAR PART OF f at $z=b$

This looks a lot like partial fractions, and in fact, for rational functions, it *is* partial fractions: if f is meromorphic with finitely many poles $b_1 \dots b_\ell$, by induction on #poles (observe: remainder $h(z)$ has one fewer pole than f), we get $f(z) = \frac{C_1(z)}{(z-b_1)^{m_1}} + \dots + \frac{C_\ell(z)}{(z-b_\ell)^{m_\ell}} + g(z)$, $C_i(z)$ polynomials of degree $< m_i$, where $g(z)$ is now analytic everywhere. What if there's ∞ many poles?