Munkres ch. 2 (§12–22) Lec. 2

(1) Topological spaces: $(X, \mathcal{T})$, $\mathcal{T} = \{U \subset X \mid U \text{ open}\}$.

Axioms: $\emptyset$, X, arbitrary unions, finite intersections of open subsets are open.

$F \subset X$ closed $\Leftrightarrow$ F^c open.

- Basis for a topology:
  - open sets = unions of elements of $\mathcal{B}$. U open $\Leftrightarrow \forall x \in U \ \exists B \in \mathcal{B}$ st. $x \in B \subset U$.
  - axioms for basis: $\bigcup \mathcal{B} = X$; $x \in B_1 \cap B_2 \Rightarrow \exists B' \in \mathcal{B}$ st. $x \in B' \subset B_1 \cap B_2$.
  - Ex: open balls $B_r(x)$ in a metric space (X, d) basis for the metric topology.
- if $\mathcal{T} \subset \mathcal{T}'$ say $\mathcal{T}'$ finer / $\mathcal{T}$ coarser.
- $f: X \to Y$ is continuous if $\forall U \subset Y$ open, $f^{-1}(U) \subset X$ is open.
  (for metric spaces, this is $\Leftrightarrow$ $\forall p \in X, \forall \varepsilon > 0, \exists \delta > 0$ st. $d_X(p,x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon$).

Lec. 3
- closure: $\bar{A} = \bigcap$ all closed subsets $\supset A$, interior $\mathrm{int}(A) = \bigcup$ all open subsets $\subset A$.
  $\bar{A} = A \cup \{\text{limit pts of } A\}$. $x \in \bar{A} \Leftrightarrow$ every nbd of x intersects A.
- limit points of subsets (x limit pt of A $\Leftrightarrow \forall U \ni x$ neighborhood, $(U - \{x\}) \cap A \neq \emptyset$)
  $\neq$ limits of sequences ($x_n \to x \Leftrightarrow \forall U \ni x$ neighborhood, all but finitely many $x_n \in U$.)
- subspace topology on $A \subset X$: $\{U \cap A \mid U \in \mathcal{T}_X\}$

Lec. 4
- product topology: basis $\left\{ \prod_{i \in I} U_i \mid U_i \subset X_i \text{ open}, \ U_i = X_i \text{ for all but finitely many } i \right\}$
  ↑ if omit this, get box topology (finer).

  For products of metric spaces, the uniform topology ($d_\infty(\vec{x}, \vec{y}) = \sup d_i(x_i, y_i)$ up to truncation) is inbetween.

  $f = (f_i): Z \to X = \prod X_i$ is continuous in product top. iff each $f_i = \pi_i \circ f: Z \to X_i$ is continuous.

Lec. 9–10
- quotient topology on $Y = X/\sim$: $U \subset Y$ is open $\Leftrightarrow q^{-1}(U) = \{x \in X \mid [x] \in U\}$ is open in X.
  $\bar{f}: Y \to Z$ continuous $\Leftrightarrow f = \bar{f} \circ q: X \to Z$ continuous & compatible with $\sim$ ($[x] = [x'] \Rightarrow f(x) = f(x')$).

Lec. 8–9, Munkres ch. 4 §30–34
- X is Hausdorff if $\forall x \neq y$, $\exists U \ni x, V \ni y$ open st. $U \cap V = \emptyset$.
  Stronger separation axioms (regular, normal) separate points from closed sets / closed sets from each other by disjoint opens. Metric spaces are normal ($\Rightarrow$ Hausdorff).
  Urysohn's thm: normal (or regular) spaces with countable basis are metrizable.

Munkres ch. 3 §23–24, 26–29

(2) Connectedness & compactness:

- X is connected if $X = U \cup V$, U, V open disjoint $\Rightarrow$ one is X and the other is $\emptyset$.

Lec. 5
- $f: X \to Y$ continuous, X connected $\Rightarrow f(X)$ connected. ($\Rightarrow$ intermediate value theorem) (connected subsets of $\mathbb{R}$ are intervals).
- path-connected := any two points of X can be joined by a path $f: I \to X$.
  path-conn. $\Rightarrow$ connected ($\not\Leftarrow$ in general)

Lec. 6
- X is compact if $\forall$ open cover $X = \bigcup_{i \in I} U_i$, $\exists$ finite subcover $X = U_{i_1} \cup \dots \cup U_{i_n}$.
- $f: X \to Y$ continuous, X compact $\Rightarrow f(X)$ compact ($\Rightarrow$ extreme value thm).
- X compact, $F \subset X$ closed $\Rightarrow F$ compact. $K \subset X$ Hausdorff, K compact $\Rightarrow K$ closed.
  in $\mathbb{R}^n$, compact $\Leftrightarrow$ closed and bounded.
- (finite) products of $\left\{ \begin{matrix} \text{compact} \\ \text{connected} \end{matrix} \right\}$ spaces are $\left\{ \begin{matrix} \text{compact} \\ \text{connected} \end{matrix} \right\}$.

Lec.7 • If (X,d) metric space & compact then:

- every open cover $X = \bigcup U_i$ has a Lebesgue number $\delta > 0$: $\mathrm{diam}(A) < \delta \Rightarrow \exists i$ st. $A \subset U_i$.
- every continuous function $f:(X,d) \to (Y,d_Y)$ is uniformly continuous

• For metric spaces, compact (open covers have finite subcovers) $\Leftrightarrow$ limit point compact (every infinite subset has a limit point) $\Leftrightarrow$ sequentially compact (every sequence has a convergent subsequence)
(always $\Rightarrow$) (always $\Leftarrow$).

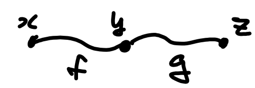
Lec.8 • A one-point compactification of X is a compact space Y st. $Y - \{\infty\} \underset{\text{homeo}}{\simeq} X$.
Build: $Y = X \cup \{\infty\}$, T_Y = opens of X + complements of compact subsets of X.
If X is locally compact ($\forall x \in X\ \exists$ nbd. $U \ni x$ and compact $C \supset U \ni x$) and Hausdorff then Y is Hausdorff and unique up to homeo.

Munkres ch.9 §51-55 +58-60

③ Homotopy and fundamental group:

• homotopy: $f_0, f_1 : X \to Y$ continuous: a homotopy is $F: X \times I \to Y$ continuous, $F_{|X\times 0} = f_0$, $F_{|X \times 1} = f_1$.

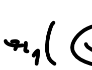
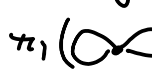
Lec.10 • paths $f_0, f_1 : I \to Y$ are path-homotopic if $\exists$ homotopy $F: I \times I \to Y$ fixing end points.

• path-homotopy classes of paths in X form a groupoid for path composition $f * g$ 

Lec.11 • —— " —— loops based at x_0 (= paths $x_0 \to x_0$) = fundamental group $\pi_1(X, x_0)$
(product = composition, identity = constant loop, inverse = reverse loop)

• $x_0, x_1 \in$ same path-component of $X \Rightarrow \pi_1(X,x_0) \cong \pi_1(X,x_1)$ (by attaching path $\alpha * f * \alpha^{-1}$).

• $f:(X,x_0) \to (Y,y_0)$ induces homomorphism $f_*: \pi_1(X,x_0) \to \pi_1(Y,y_0)$. Functorial ($(f\circ g)_* = f_* \circ g_*$).

Lec.13 • Ex: $\mathbb{R}^n$, convex subsets of $\mathbb{R}^n$, S^n $n \geq 2$ are simply connected ($\pi_1 = \{1\}$).
$\pi_1(S^1, b_0) \simeq \mathbb{Z}$, $\pi_1($  $) \simeq \mathbb{Z}^2$, $\pi_1($  $) \simeq$ free group $\langle a, b \rangle$

• Applications of $\pi_1(S^1) = \mathbb{Z}$:
- $\nexists$ retraction $r: B^2 \to S^1$ (ie. r continuous, $r_{|S^1} = id_{S^1}$).
- every continuous $f: B^2 \to B^2$ has a fixed pt ($f(x) = x$) (Brouwer)

Lec.10 • Deformation retraction: $r: X \to A$ retraction ($r_{|A} = id_A$) st. $i \circ r$ is homotopic to id_X among maps that leave A fixed. ie. $H: X \times I \to X$, $H(x,0) = x$, $H(x,1) \in A\ \forall x \in X$, $H(a,t) = a\ \forall a \in A$
Then $\pi_1(A, a_0) \simeq \pi_1(X, a_0)$ (i_*, r_* inverse isoms.)

• The same holds more generally for homotopy equivalences $X \underset{g}{\overset{f}{\rightleftarrows}} Y$, $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$.

Lec.12 • Covering spaces: $p: E \to B$, $\forall b \in B\ \exists U \ni b$ evenly covered by p
($p^{-1}(U)$ = disjoint union of slices V_α, each $p_{|V_\alpha}: V_\alpha \underset{\text{homeo}}{\xrightarrow{\sim}} U$).

• every path $f: I \to B$ starting at b_0 has unique lift $\tilde{f}: I \to E$ starting at $e_0 \in f^{-1}(b_0)$. (Path) homotopies lift to (path) homotopies.

• Looking at end points of lifts of loops in (B, b_0), get lifting map $\pi_1(B, b_0) \to p^{-1}(b_0)$.

③

- Those loops which lift to a loop in (E, e_0) form a subgroup $H \subset \pi_1(B, b_0)$, and $p_*: \pi_1(E, e_0) \xrightarrow{\sim} H \subset \pi_1(B, b_0)$.

Lec. 14

- A map $g: (Y, y_0) \to (B, b_0)$ lifts to $\tilde{g}: (Y, y_0) \to (E, e_0)$ iff $g_*(\pi_1(Y, y_0)) \subset H$.

Munkres §79-80

- Classification of covering spaces (up to equivalence) $\Leftrightarrow$ classif. of subgroups $H \subset \pi_1(B)$ (up to conjugacy).
- Universal cover: simply-connected E (ie., $H = \{1\}$).

Lec. 15
Munkres §70

- <u>Van Kampen</u>: $X = U \cup V$, U, V open, $U \cap V \ni x_0$ path-connected $\Rightarrow$
    - $\pi_1(X, x_0)$ is generated by the images of $j_{1*}: \pi_1(U) \to \pi_1(X)$, $j_{2*}: \pi_1(V) \to \pi_1(X)$
    - if $\pi_1(U \cap V) = \{1\}$ then $\pi_1(X)$ is the free product $\pi_1(U) * \pi_1(V)$
    - otherwise, quotient by smallest normal subgroup that makes $i_{1*}(g) = i_{2*}(g)$ $\forall g \in \pi_1(U \cap V)$.