Rudin ch. 3
McMullen §4
Lec. 16

(1) Sequences and series

- in (X,d) metric space, $x_n \to x$ iff $d(x_n,x)\to 0$, ie. $\forall \varepsilon\ \exists N$ st. $\forall n\geq N$, $d(x_n,x)<\varepsilon$.
- (x_n) Cauchy sequence $:= \forall\varepsilon\ \exists N$ st. $\forall m,n\geq N$, $d(x_n,x_m)<\varepsilon$.
- convergent $\Rightarrow$ Cauchy ; $\Leftarrow$ if (X,d) is complete (eg. $\mathbb{R}, \mathbb{R}^n, \mathbb{C}$; $C^0([a,b])$ with sup norm, compact metric spaces, ...)
- compactness of $[-M,M]$ $\Rightarrow$ every bounded seq. in $\mathbb{R}, \mathbb{R}^n, \mathbb{C}$ has a convergent subsequence.
- a monotonic sequence in $\mathbb{R}$ (eg. $a_n \leq a_{n+1}$) converges iff it is bounded ($\Rightarrow \lim_{n\to\infty} a_n = \sup\{a_n\}$).
- $a_n \to +\infty$ means $\forall M\ \exists N$ st. $n\geq N \Rightarrow a_n > M$. ($\Leftrightarrow$ converges in $\mathbb{R}\cup\{\pm\infty\}$)
- if (a_n) bounded then $\limsup a_n :=$ largest limit of a convergent subsequence of (a_n). Similarly liminf.
- series $\sum_{n=0}^{\infty} a_n$ converges iff partial sums $s_n = \sum_{k=0}^{n} a_k$ are a convergent sequence.
- $\sum a_n$ converges $\Rightarrow a_n \to 0$. For $a_n \geq 0$, $\sum a_n$ converges iff partial sums are bounded.
- comparison criterion : $0\leq a_n \leq b_n$, $\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent
- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ converges iff $|x|<1$, $\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ converges iff $\alpha>1$
- $\sum a_n$ converges absolutely if $\sum |a_n|$ converges ; abs. conv. $\Rightarrow$ convergent.
  but not $\Leftarrow$, eg. alternating series ($\sum (-1)^n a_n$, $a_n \searrow 0$) always converge.
- Root test : $\limsup |a_n|^{1/n} < 1 \Rightarrow \sum a_n$ converges (absolutely), $>1 \Rightarrow$ diverges.

Rudin ch. 4 and 7
McMullen §3
Lec. 16

(2) Continuous real functions of 1 variable

- Continuity at x ($\forall\varepsilon>0\ \exists\delta$ st. $\forall y$, $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$) $\Leftrightarrow \lim_{t\to x} f(t) = f(x)$.
  Infinite limits, limits at ∞ = work in $\mathbb{R}\cup\{\pm\infty\}$.
- Compactness of $[a,b]$ $\Rightarrow$ continuous functions on $[a,b]$ are uniformly continuous, ie.
  $\forall\varepsilon\ \exists\delta / \forall x,y$, $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$ (same δ works $\forall x$).
- intermediate value theorem $f([a,b])$ is connected $\Rightarrow$ contains all reals between $f(a)$ & $f(b)$.
- extreme value theorem $f([a,b])$ is compact $\Rightarrow$ bounded and contains its inf & sup.
- $f_n \to f$ pointwise if $\forall x$, $f_n(x)\to f(x)$. (= product topology)
- $f_n \to f$ uniformly if $\|f_n - f\|_\infty = \sup_x |f_n(x)-f(x)| \to 0$.
- if f_n is continuous and $f_n \to f$ uniformly then f is continuous
  ie. $C^0 \subset \{\text{functions}\}$ is closed in uniform topology ; $(C^0, \|\cdot\|_\infty)$ is complete..
- uniform Cauchy criterion for unif. convergence $\rightsquigarrow$ Weierstrass M-test for series :
  if $\sup |f_n| \leq M_n$ and $\sum M_n$ converges then $\sum f_n$ converges uniformly.
- Power series : $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Radius of convergence : $R = \dfrac{1}{\limsup |a_n|^{1/n}} \in [0,\infty]$
  series converges for $|z|<R$, uniformly on $\{|z|\leq r\}\ \forall r<R$, diverges for $|z|>R$.
  f is continuous on $\{|z|<R\}$ (... and differentiable to all orders, see below)

Rudin ch. 5
McMullen §5
Lec. 17

(3) Derivatives in 1 real variable : $f'(x) = \lim_{t\to x} \dfrac{f(t)-f(x)}{t-x}$

- differentiable $\Rightarrow$ continuous

- mean value thm: $f:[a,b]\to\mathbb{R}$ differentiable $\Rightarrow \exists c\in(a,b)$ st. $f(b)-f(a)=f'(c)(b-a)$.
- Taylor's thm: f n times differentiable $\Rightarrow \exists c\in(a,b)$ st. $f(b)=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(b-a)^k+\frac{f^{(n)}(c)}{n!}(b-a)^n$
- most C^∞ functions cannot be expressed as power series (Taylor series $\not\to f$).
- $f_n\in C^1$, $f_n\to f$ pointwise, $f_n'\to g$ uniformly $\Rightarrow f\in C^1$, $f'=g$, and $f_n\to f$ in C^1 top.
- $C^k([a,b],\mathbb{R})=\{f$ k times diff$^{\text{able}}$, $f^{(k)}$ continuous$\}$, $\|f\|_{C^k}=\sum_{j=0}^{k}\|f^{(j)}\|_\infty$ is a complete metric space.
- $f(x)=\sum_{n=0}^\infty a_n x^n$ power series $\Rightarrow f(x)$ is C^∞ on $(-R,R)$, and $f'(x)=\sum n a_n x^{n-1}$.

Rudin ch. 6
McMullen §6

④ Riemann integral:

- $f:[a,b]\to\mathbb{R}$ bounded, $a=x_0<x_1<\dots<x_n=b \Rightarrow s_i=\inf f([x_{i-1},x_i])$, $S_i=\sup f([x_{i-1},x_i])$,
  lower/upper Riemann sums: $I_-(f)=\sup\{\sum s_i(x_i-x_{i-1})\}$, $I_+(f)=\inf\{\sum S_i(x_i-x_{i-1})\}$

  Lec. 18

  f is Riemann integrable on $[a,b]$ if $I_-(f)=I_+(f)$ $\left(\underset{\text{def}}{=}\int_a^b f(x)\,dx\right)$.
- $f\le g \Rightarrow \int_a^b f\,dx\le\int_a^b g\,dx$; $a<c<b \Rightarrow \int_a^b=\int_a^c+\int_c^b$; etc.
- f (piecewise) continuous on $[a,b]$ $\Rightarrow$ integrable.
- if $f\in C^0([a,b])$ then $F(x)=\int_a^x f(t)\,dt$ is differentiable and $F'=f$ (fund. thm. calc.)
- $\left|\int_a^b f\,dx-\int_a^b g\,dx\right|\le\int_a^b|f-g|\,dx\le(b-a)\|f-g\|_\infty$. $\Rightarrow$ if $f_n\to f$ uniformly then $\int_a^b f_n\,dx\to\int_a^b f\,dx$
- the L^p norm: $\forall p\ge1$, $\|f\|_{L^p}=\left(\int_a^b|f(x)|^p dx\right)^{1/p}$ coarser than uniform topology ($f_n\to f$ uniformly on $[a,b]$ $\underset{\not\Leftarrow}{\Rightarrow}$ $f_n\to f$ in L^p)

  L^2 inner product: $\langle f,g\rangle_{L^2}=\int_a^b fg\,dx$

Rudin ch. 7–8
McMullen §7

⑤ More about C^0 functions

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- $\mathcal{F}\subset C^0(K)$ is (unif.) equicontinuous if $\forall\varepsilon>0$ $\exists\delta>0$ st. $\forall f\in\mathcal{F}$, $\forall x,y\in K$, $d(x,y)<\delta\Rightarrow d(f(x),f(y))<\varepsilon$
  (K: compact metric space; δ: indep^t of $x\in K$ and $f\in\mathcal{F}$)
- Arzela-Ascoli: if $\{f_n\}\in C^0(K)$ uniformly bounded for $\|\cdot\|_\infty$ and equicontinuous then $\exists$ uniformly convergent subsequence.
  $\mathcal{F}\subset(C^0(K),\|\cdot\|_\infty)$ is compact iff it is closed, bounded, and equicontinuous.

Lec. 19

- Weierstrass thm: polynomials are dense in $C^0([a,b],\mathbb{R})$, ie. $\forall f\in C^0$ $\exists P_n\in\mathbb{R}[x]$ st. $P_n\to f$ unif. on $[a,b]$.
  Proof uses convolution $(f*g)(x)=\int_{-\infty}^{\infty}f(x-t)g(t)\,dt$ of f with suitable polynomials.
- Stone-Weierstrass: K compact metric space, $\mathcal{A}\subset C^0(K)$ algebra ($f,g\in\mathcal{A}\Rightarrow f+g,\ cf,\ fg\in\mathcal{A}$), (in $\mathbb{C}$-valued case: $f\in\mathcal{A}\Rightarrow\bar f\in\mathcal{A}$), separating points ($\forall a\ne b$ $\exists f,g\in\mathcal{A}$ st. $f(a)=1$, $f(b)=0$; $g(a)=0$, $g(b)=1$).
  $\Rightarrow\mathcal{A}$ is dense in $(C^0(K),\|\cdot\|_\infty)$
- Fourier series of $f:\mathbb{R}\to\mathbb{C}$ (2π-periodic): $\sum_{n\in\mathbb{Z}}c_n e^{inx}$, where $c_n(f)=\frac{1}{2\pi}\int_0^{2\pi}e^{-inx}f(x)\,dx$.
  Trigonometric polynomials are dense in $(C^0(S^1,\mathbb{C}),\|\cdot\|_\infty)$ hence in L^2 norm.
  The Fourier sum $s_n(f)=\sum_{-n}^{n}c_k e^{ikx}$ = closest (in L^2-dist.) approximation of f by trig. poly.

$\Rightarrow$ Parseval: $\forall f\in C^0$, $\|s_n-f\|_{L^2}^2=\frac{1}{2\pi}\int|f(x)-s_n(x)|^2\,dx\to0$; $\sum_{n\in\mathbb{Z}}|c_n|^2=\frac{1}{2\pi}\int|f|^2dx$ converges.

Lec. 20 • Dirichlet: $\| f \in C^1 \Rightarrow s_n \to f$ uniformly. (uses: $s_n = f * D_n$ convolve by Dirichlet kernel)
(not necessarily true for $f \in C^0$; however, Fejer: $f \in C^0 \Rightarrow \frac{s_0 + \dots + s_{n-1}}{n} \to f$ uniformly)

Rudin ch. 9
McMullen §8

⑥ Differentiation in several variables

- $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ differentiable at $x \in U$: $\exists Df(x) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ st. $f(x+v) = f(x) + Df(x)v + o(|v|)$
- the matrix of $Df(x)$ has entries $\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ operator norm $\|Df(x)\| = \sup_{v \neq 0} \frac{|Df(x)v|}{|v|}$
- $f \in C^1$ if $Df: U \to \mathbb{R}^{m \times n}$ is C^0. ($\Leftrightarrow$ partial derivatives exist and are C^0). (false without continuity)
- chain rule: $D(f \circ g)(x) = Df(g(x)) \circ Dg(x)$.

Lec. 21

- mean value inequality: $|f(b) - f(a)| \le |b-a| . \sup_{x \in [a,b]} \|Df(x)\|$.
- if $f \in C^2$ then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$.
- inverse function thm: if $f \in C^1$ and $Df(x_0): \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then $\exists$ nbd $U \ni x_0$ st. $f_{|U}: U \xrightarrow{\sim} f(U)$ is a diffeomorphism (ie. bijection, f & f^{-1} both C^1).
- implicit function thm: $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $(x,y) \mapsto f(x,y)$ differentiable, $Df = Df_x \oplus Df_y : \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^m$.

  If $f(x_0, y_0) = 0$ and $Df_y(x_0, y_0)$ is invertible then $\exists$ nbd. $U \ni x_0$, $V \ni y_0$ st.
  $\forall x \in U\ \exists!\ y = g(x) \in V$ st. $f(x,y) = f(x, g(x)) = 0$; g is diff-able, $Dg = -(Df_y)^{-1} \circ Df_x$.

eg. a hypersurface $S = \{f(x_1 \dots x_n) = 0\}$ (f C^1, $Df(x) \neq 0$ $\forall x \in S$) is locally a graph $x_i = g((x_j)_{j \neq i})$.

Rudin ch. 10
McMullen §9
Lec. 22

⑦ Integration in several variables, differential forms

- $D \subset \mathbb{R}^n$ product of intervals (or domain with piecewise smooth boundary), f (piecewise) C^0
  $\Rightarrow \int_D f\, dx_1 \dots dx_n = \int_D f\, |dx| = \begin{cases} \text{iterated integral (in any order) (Fubini's thm)} \\ \text{Riemann } \int \text{ splitting } D \text{ into small cubes and bounding } f \text{ by its inf/sup on each cube.} \end{cases}$
- change of variables: φ diffeomorphism, $f \in C^0 \Rightarrow \int_{\varphi(U)} f(y)\, d|y| = \int_U f(\varphi(x))\, |\det D\varphi(x)|\, d|x|$.
- differential forms: 1-form $\omega = \sum_i p_i(x)\, dx_i : \mathbb{R}^n \supset U \to T^*$, $\omega(x)(v) = \sum p_i(x) v_i$

  $\gamma: [0,1] \to \mathbb{R}^n$, piecewise C^1, $\int_\gamma \omega = \int_0^1 \omega(\gamma(t))\left(\frac{d\gamma}{dt}\right) dt = \int_0^1 \left(\sum_i p_i(\gamma(t)) \frac{d\gamma_i}{dt}\right) dt$
- k-forms: $\omega = \sum_{i_1 < \dots < i_k} p_I(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_k} : \mathbb{R}^n \supset U \to \Lambda^k T^*$ $\omega(x)(v_1, \dots, v_k) \in \mathbb{R}$. alternating multilinear form

  $\omega \in \Omega^k(U) = C^\infty(U, \Lambda^k T^*)$

  $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

  $\wedge : \Omega^k \times \Omega^l \to \Omega^{k+l}$ $(f\, dx_I) \wedge (g\, dx_J) = (fg)\, dx_I \wedge dx_J = \begin{cases} \pm (fg)\, dx_{I \cup J} & I \cap J = \phi \\ 0 & I \cap J \neq \phi \end{cases}$

  $d: \Omega^k \to \Omega^{k+1}$ $d\left(\sum_I p_I\, dx_I\right) = \sum_{I,j} \frac{\partial p_I}{\partial x_j} dx_j \wedge dx_I$.

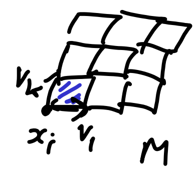
  $d^2 = 0$. ω is closed if $d\omega = 0$, exact if $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}$.

- Poincaré lemma: on $U \subset \mathbb{R}^n$ convex, $\underset{\text{closed}}{d\omega = 0} \Leftrightarrow \exists \alpha$ st. $\underset{\text{exact}}{\omega = d\alpha}$
  (in general, $\ker d / \operatorname{Im} d$ = De Rham cohomology, depends on alg. top. of U).

Lec. 23

- <u>Pullback</u>: $\varphi : U \to V$ smooth map $\rightsquigarrow \varphi^* : \Omega^k(V) \to \Omega^k(U)$
  $$(\varphi^*\omega)(x)(v_1, \dots, v_k) = \omega(\varphi(x))(D\varphi(x)v_1, \dots, D\varphi(x)v_k)$$
  - $\varphi^*(f) = f \circ \varphi$, and $\varphi^*(dy_j) = d(y_j \circ \varphi) = \sum \frac{\partial \varphi_j}{\partial x_i} dx_i = d\varphi_j$.
    $\Rightarrow \varphi^*\left(\sum_J p_J(y)\, dy_{j_1} \wedge \dots \wedge dy_{j_k}\right) = \sum_J p_J(\varphi(x))\, d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}$
  - $\varphi^*(d\omega) = d(\varphi^*\omega)$.

- <u>Integration</u>: $\omega \in \Omega^k$, M k-dimensional parametrized by $M = \varphi(D)$, $D \subset \mathbb{R}^k$



  $$\int_M \omega = \lim \sum_i \omega(x_i)(v_1, \dots v_k) \quad \text{splitting } M \text{ into small grid parallelepipeds}$$
  $$= \int_D \omega(\varphi(t))\left(\frac{\partial \varphi}{\partial t_1}, \dots, \frac{\partial \varphi}{\partial t_k}\right) dt_1 \dots dt_k = \int_D \varphi^*\omega$$

- general <u>pullback formula</u>: $\varphi : \underset{\mathbb{R}^m}{U} \to \underset{\mathbb{R}^n}{V}$, $\omega \in \Omega^k(V)$, $\underset{k\text{-dim.}}{M \subset U} \Rightarrow \int_{\varphi(M)} \omega = \int_M \varphi^*\omega$.
  ($\Leftrightarrow$ change of var's / chain rule)

- <u>Stokes' theorem</u>: M k-dim'l, $\omega \in \Omega^{k-1} \Rightarrow \int_M d\omega = \int_{\partial M} \omega$.