Riemann surfaces were historically introduced to deal with the multivalued nature of certain algebraic functions and their integrals.

* Eg. to evaluate $\int_{z_0}^{z_1} \frac{dz}{\sqrt{z^2+1}}$, you might use trig. substitutions (here: $z=\sinh(u)$)

but a more elegant approach is to think of this as a path integral on a Riemann surface: since $\sqrt{z^2+1}$ isn't single-valued (2 choices whenever $z \notin \{\pm i\}$), its graph is rather a 2-sheeted covering space of $\mathbb{C} \setminus \{\pm i\}$, $w = \pm\sqrt{z^2-1}$. If we vary z along eg. a circle around one of $\pm i$, the lift of this path to the covering changes sheets: starting at w we come back to $-w$.

So: we introduce $\Sigma = \{(z,w) \in \mathbb{C}^2 / w^2 = z^2+1\}$ and now view z and w as single-valued analytic functions on Σ rather than multivalued functions on $\mathbb{C}$.

Σ is an example of a complex manifold – near each point of Σ we can use one of w or z as local coordinate and express all quantities as analytic functions of it.

In particular, our integral is now best understood as $\int_{p_0}^{p_1} \frac{dz}{w}$ between points $p_0, p_1 \in \Sigma$. $p_0 = (z_0, w_0)$, $p_1 = (z_1, w_1)$.

This is sometimes a pointless complication if you already had a clear mind about what to do with the integral but in general it can bring considerable insight.

* Here, the remarkable fact is that Σ is biholomorphic to a domain in the complex plane. Explicitly, in terms of the Riemann sphere $S = \mathbb{C} \cup \{\infty\}$, we have inverse analytic bijections

$$S - \{\pm 1\} \xrightarrow{\cong} \Sigma = \{(z,w) \mid w^2 = z^2+1\}$$
$$\lambda \longmapsto \left(\frac{2\lambda}{1-\lambda^2}, \frac{1+\lambda^2}{1-\lambda^2}\right)$$
$$\frac{w-1}{z} \longleftarrow (z,w)$$

So: we can transform our path integral on Σ into one on $S - \{\pm 1\}$, by the change of variables $w = \frac{1+\lambda^2}{1-\lambda^2}$, $dz = d\left(\frac{2\lambda}{1-\lambda^2}\right) = \frac{2(1+\lambda^2)}{(1-\lambda^2)^2}\, d\lambda$.

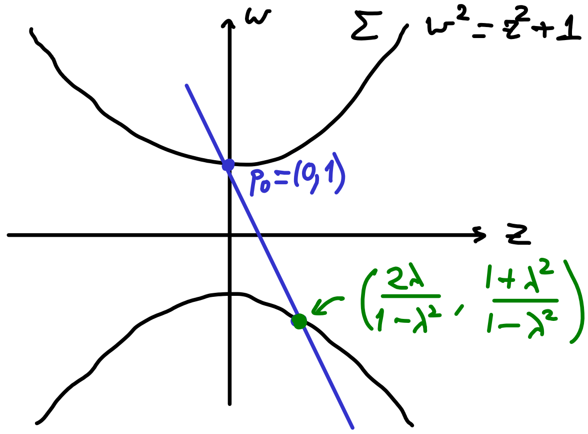
So $\int_{p_0}^{p_1} \frac{dz}{w} = \int_{\lambda_0}^{\lambda_1} \frac{2\, d\lambda}{1-\lambda^2}$ which is now easy to deal with by partial fractions.

* What's the geometry behind this change of variables – how does one come up with it?

$\Sigma = \{(z,w) \in \mathbb{C}^2 / w^2 = z^2+1\}$ is an algebraic equation of degree 2, so its intersection with a (complex) line in $\mathbb{C}^2$ consists of (usually) 2 points.

So: we can project it along the family of lines through a point $p_0 \in \Sigma$ (here $(0,1)$), each of these lines meets Σ at p_0 and (usually) one other point.

(Compare w/ the idea of stereographic projection! $S^2 \subset \mathbb{R}^3$ degree 2 eq^n; it's the same conceptual idea, in $\mathbb{C}^2$ instead of $\mathbb{R}^3$)



L_λ line with slope λ through $(0,1)$:
$w = \lambda z + 1$.

The line of slope λ through p_0 has eq^n $w = \lambda z + 1$

Plugging this into $w^2 = z^2+1$ gives a degree 2 equation in z (with coeff^{ts} depending on λ), which always has $z=0$ as one of its roots, so it's especially easy to find the other root!

$$(\lambda z+1)^2 = z^2+1$$
$$\rightsquigarrow (\lambda^2-1)z^2 + 2\lambda z = 0$$
$$\to z=0 \text{ or } z = \frac{2\lambda}{1-\lambda^2}.$$

Also, every point $p \in \Sigma$ $(p \neq p_0)$ arises from this construction, by taking the line $(p_0 p)$.

(Special cases: • for $\lambda = 0$, L_λ is tangent to Σ at p_0 so we get double root $z=0$
• for $\lambda = \pm 1$, the other intersection of L_λ and Σ goes missing ("at ∞")
• to obtain the point $(0,-1) \in \Sigma$, need to allow slope $\lambda = \infty$).

$\rightsquigarrow$ this gives a bihol^m. $S - \{\text{finite set}\} \xrightarrow{\sim} \Sigma$ given by rational functions
(say Σ is a rational curve; "curve" because it's complex 1-dimensional)
even though it looks actually like a surface... (real 2-dimensional).

- This process allows us to evaluate path integrals on algebraic curves $\Sigma \subset \mathbb{C}^2$ defined by any quadratic polynomial $Q(z,w) = 0$... but then it breaks down.

Q: calculate the arclength of a portion of ellipse $x^2 + \frac{y^2}{2} = 1$ between (x_0,y_0) & (x_1,y_1)

If you write $y = \pm\sqrt{2(1-x^2)}$ and use $\int_{x_0}^{x_1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ you end up with something like $\int_{x_0}^{x_1} \sqrt{\frac{1+x^2}{1-x^2}}\,dx$. If you instead use parametric length, you end up with $\int_{\theta_0}^{\theta_1} \sqrt{1+\cos^2\theta}\,d\theta$.

Massaging this further we can reduce to e.g. $\int \frac{dx}{\sqrt{1-x^4}}$

But none of these "elliptic integrals" can be expressed in terms of known functions.

So early 19th century mathematicians remained stuck until Riemann, Abel, ... provided the right viewpoint – Riemann surfaces are <u>needed</u> to make sense of what's going on.

(This is a topic at the intersection of complex analysis, topology, and algebraic geometry!)

So we now look at the graph of $\sqrt{1-z^4}$: $\Sigma = \{(z,w) \in \mathbb{C}^2 / w^2 = z^4 - 1\}$

<u>Claim</u>: the reason this one is so different from the previous one is that it's not an open subset of the Riemann sphere, but rather an open subset of a torus (an "<u>elliptic curve</u>" - the name comes from the problem of elliptic integrals & has stuck)

<u>How do we see this</u>? Ans: project to the z-coordinate : $(z,w) \mapsto z$.

This map is a "branched covering" - a two-sheeted covering map after we remove the roots p_i of the polynomial in z (here $z^4-1 \Rightarrow \pm 1, \pm i$) from $\mathbb{C}$, and $q_i = (p_i, 0)$ from Σ.

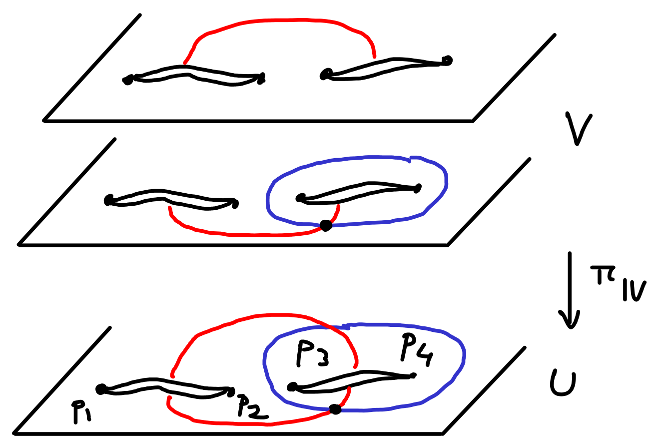
$\Rightarrow$ the map $\Sigma \setminus \{q_i\} \xrightarrow{\pi} \mathbb{C} \setminus \{p_i\}$ is a <u>2:1 covering</u>.
$\qquad (z,w) \longmapsto z$

The p_i are <u>branch points</u>: the lift of a small circle around p_i is a path that ends up on the opposite sheet of where it started $(w \mapsto -w)$; in general, a loop in $\mathbb{C} - \{p_i\}$ lifts to a loop in $\Sigma - \{q_i\}$ iff the sum of its winding numbers around $p_1 \ldots p_4$ is even.

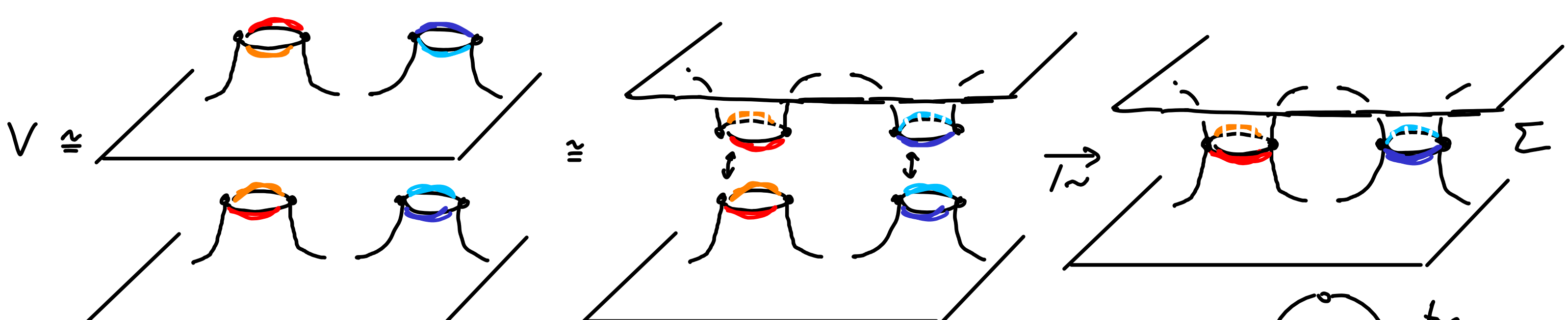
Draw two arcs γ, γ' in $\mathbb{C}$ connecting p_1 to p_2 and p_3 to p_4 (for example), and let $U = \mathbb{C} \setminus (\gamma \cup \gamma')$. Then any loop in U has even total winding number, so lifts to a loop in Σ. Hence the restricted covering map from $V = \pi^{-1}(U)$ to U is trivial: $V = V_+ \sqcup V_-$, $\pi_{|V_\pm} : V_\pm \xrightarrow{\sim} U$.

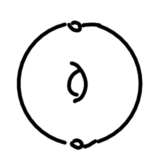
making the slits in these planes more visible:

Adding back in the missing arcs $\gamma \cup \gamma'$, the lift of a path in $\mathbb{C}$ jumps between the two sheets $V_\pm$ each time it crosses $\gamma \cup \gamma'$, so Σ is obtained from V by attaching one side of each slit in each sheet to the other side of the same slit in the other plane.



To picture this: flip one sheet & glue



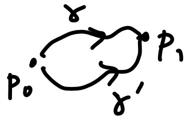
≅  torus minus 2 points.

(can compactify $\mathbb{C}$ to S and Σ to a torus $\bar{\Sigma}$ by adding 2 preimages of ∞).

The implication for complex analysis is that, since $\bar{\Sigma}$ isn't simply connected, path integrals on it depend on the path of integration.

Returning to our integral $\int \frac{dz}{w}$ on $\Sigma = \{w^2 = z^4 - 1\}$
(or another polynomial of deg. 3 or 4) with simple roots

- $\frac{dz}{w}$ is actually an analytic 1-form on $\overline{\Sigma}$, without poles or zeroes
  (at $(z,w) = (p_i, 0)$, the local coordinate on Σ is actually w, not z,
  but $w^2 = P(z) \rightsquigarrow 2w\,dw = P'(z)\,dz \Rightarrow \frac{dz}{w} = \frac{2\,dw}{P'(z)}$ no pole).
- the integral $\int_{p_0}^{p_1} \frac{dz}{w}$ is invariant under path homotopy (Cauchy) but depends on homotopy class.

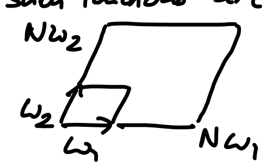
Choose loops α_1, α_2 generating $\pi_1(\overline{\Sigma}) \simeq \mathbb{Z}^2$, then a change of homotopy class modifies the value of $\int$ by an integer linear combination of the <u>periods</u> $\omega_1 = \int_{\alpha_1} \frac{dz}{w}$, $\omega_2 = \int_{\alpha_2} \frac{dz}{w}$:
given 2 paths 

, $[\gamma - \gamma'] = m_1[\alpha_1] + m_2[\alpha_2]$ for some $m_1, m_2 \in \mathbb{Z}$

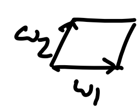
$$\Rightarrow \int_\gamma - \int_{\gamma'} = m_1\omega_1 + m_2\omega_2.$$

- $\int_{p_0}^{p} \frac{dz}{w} = F(p)$ defines an analytic mapping $\overline{\Sigma} \xrightarrow{F} \mathbb{C}/\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ which
  - cannot be expressed in terms of elementary functions
  - has everywhere nonzero derivative, so F is a local homeomorphism, and in fact a covering map.
  - by winding number arguments (if you're a complex analyst) or studying the map on fundamental groups (if you're a topologist), $\#F^{-1}(c) = 1\ \forall c$, so in fact F is a <u>biholomorphism</u>.
- What is the inverse of F? <u>Ans:</u> a doubly periodic function! ($\approx$ Weierstrass $\wp$-function)

<u>The Weierstrass $\wp$-function</u>: look for doubly periodic functions $f(z+\omega_1) = f(z+\omega_2) = f(z)$?

If f is analytic then it's bounded hence constant, so the only interesting such functions are meromorphic. Residue formula integrating around a large parallelogram


$\Rightarrow$ Σ of residues in fundamental domain must be zero
(since path $\int$ linear in N vs. Σ Res quadratic in N)
$\Rightarrow$ can't have just a single pole of order 1 in the fundamental domain.

The simplest of all either have one pole of order 2, or 2 poles of order 1, in 

Weierstrass' starting point has a pole of order 2, with vanishing residue
$\Rightarrow$ up to translation $z \mapsto z-a$ we can place the pole at 0, polar part $\frac{1}{z^2}$.

Following our study of infinite sums and how to achieve convergence, this leads to

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \qquad (\omega = n_1\omega_1 + n_2\omega_2,\ (n_1,n_2) \in \mathbb{Z}^2 - \{(0,0)\})$$

the Weierstrass $\wp$-function.

This series converges uniformly on compact sets (using: $\sum_{\omega\neq 0} \frac{1}{|\omega|^3} < \infty$)

$$\wp'(z) = -2\sum_{\omega} \frac{1}{(z-\omega)^3}$$ is obviously periodic, so $\wp(z+\omega_1) - \wp(z) = \text{const.}$
$\wp(z+\omega_2) - \wp(z) = \text{const.}$

But clearly $\wp(z)$ is an even function $\wp(-z) = \wp(z)$ $\Rightarrow$ take $z = \frac{\omega_1}{2}$, $z = \frac{\omega_2}{2}$ in to get $\wp$ is periodic too.

Working out the Laurent expansions at $z=0$

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \dots$$ (constant term vanishes; odd terms vanish since $\wp$ even)

for some constants $g_2, g_3 \in \mathbb{C}$ (depending on ω_1, ω_2).

$$\wp'(z) = \frac{-2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + \dots$$

$$\Rightarrow \wp'(z)^2 = \frac{4}{z^6} - \frac{2g_2}{5z^2} - \frac{4g_3}{7} + \dots$$

$$\text{vs. } 4\wp(z)^3 = \frac{4}{z^6} + \frac{3g_2}{5z^2} + \frac{3g_3}{7} + \dots$$

$\Rightarrow \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$

(polar parts match, so equal up to entire periodic function = constant, but constant terms match too)

Outcome: $z \longmapsto (\wp(z), \wp'(z))$ gives a biholomorphism

$$\mathbb{C}/_{\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2} \xrightarrow{\simeq} \{(x,y) \in \mathbb{C}^2 \,/\, y^2 = 4x^3 - g_2 x - g_3\} \cup \{\infty\}$$

(another elliptic curve!)

$d\wp(z) = \wp'(z)\,dz \Rightarrow dz = \frac{d\wp(z)}{\wp'(z)} = \frac{dx}{y}$, ie. the inverse function is $\int \frac{dx}{y} = \int \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}$

(This is almost what we had in the other direction, except this one has one of the 4 branch points at ∞, unlike our previous example where all 4 $p_i \in \mathbb{C}$. Simple coordinate transformations by rational functions let us switch between the two.)

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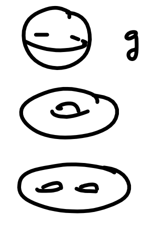
* One more neat fact to end with: consider $f(x,y) \in \mathbb{Q}[x,y]$ polynomial w/ rational coefficients.

<u>Q</u>: how many <u>rational solutions</u> $\{(x,y) \in \mathbb{Q}^2 \,/\, f(x,y) = 0\}$?

In fact the answer is governed by the <u>topology</u> of the Riemann surface $\bar{\Sigma}$ obtained by compactification of $\Sigma = \{(x,y) \in \mathbb{C}^2 \,/\, f(x,y) = 0\}$, specifically its <u>genus</u> g



$g=0$

$g=1$

$g=2$

....

If $g=0$ (rational curve, $\cong S = \mathbb{C} \cup \{\infty\}$) or 1 (elliptic $\simeq \mathbb{C}/_{\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2}$) then alg. operations (eg addition in elliptic curve) yield new rational solutions from known ones, so #sol's over $\mathbb{Q}^2$ <u>can</u> be infinite

<u>Thm</u> (Faltings) || If $g \geq 2$ then there are only finitely many rational solutions.

(At this point we've brought together algebra, analysis, topology, geometry & number theory! This is a good place to end Math 55.)