Math 55b Homework 12

Due Wednesday April 28, 2021.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.

Material covered: Residues and integration; sum and product expansions (Ahlfors 4.5, 5.1-5.2)

1. Let D is a bounded domain with (piecewise) smooth boundary $\partial D = \gamma$, $f(z)$ an analytic function on an open set containing $\overline{D}$, and assume that f does not vanish at any point of γ. Denote by z_i the zeroes of f inside D and m_i their multiplicities. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}\,dz = \sum m_i z_i.$$

2. Prove or disprove: if $f(z)$ is analytic on the unit disc D and has n zeros in D, then $f'(z)$ has at least $n-1$ zeros in D. What happens if "at least" is replaced by "at most"?

3. Use the method of residues to evaluate the integrals $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}\,dx$ and $\displaystyle\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx$.

4. Let $f(z) = \pi\cot(\pi z)$. We have seen in class that $f(z)$ has simple poles at all integers, with residues all equal to 1. Let $k \geq 1$ be a positive integer.

(a) For $n = 1, 2, \dots$, let $R_n = \{z \in \mathbb{C},\ |\mathrm{Re}(z)| \leq n + \frac{1}{2} \text{ and } |\mathrm{Im}(z)| \leq n\}$. Show that

$$\lim_{n\to\infty} \int_{\partial R_n} \frac{f(z)}{z^{2k}}\,dz = 0.$$

(Hint: do this directly, not using residues: bound the integrand over the horizontal edges by showing that $|\cot(\pi z)| \to 1$ as $|\mathrm{Im}(z)| \to \infty$, and over the vertical edges by showing that $|\cot(\pi z)|$ is uniformly bounded by a constant (in fact, by 1) for all z such that $\mathrm{Re}(z) \in \mathbb{Z} + \frac{1}{2}$.)

(b) Use the residue theorem to show that $\mathrm{Res}_{z=0}\Big(f(z)/z^{2k}\Big) + 2\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{2k}} = 0.$

(c) By calculating the Laurent series of $f(z)$ near $z = 0$, deduce the values of $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$.

5. (a) Show that $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$. (b) Show that, for $|z| < 1$, $\displaystyle\prod_{n=0}^{\infty}(1 + z^{2^n}) = \frac{1}{1-z}$.

6. (a) What is the value of $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2 + a^2}$?

(b) Optional: deduce from this the following identities (one of which has a much shorter proof):

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2}\coth(\pi) - \frac{1}{2}; \quad \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2 - \frac{1}{4}} = 0; \quad \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2 - \frac{1}{16}} = \frac{-4\pi}{\cos(2\pi z)}.$$

7. For $x \in [0,1]$ we define $f(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{t(1-t^2)}}$.

(a) Show that there exists a unique continuous complex valued function $F(z)$ on $\overline{\mathbb{H}} = \{\mathrm{Im} z \geq 0\}$ such that F is analytic on $\mathbb{H} = \{\mathrm{Im} z > 0\}$ and $F(x) = f(x)$ for all $x \in [0,1]$.

(b) Show that $S = F(\mathbb{H})$ is the interior of a square in $\mathbb{C}$, and that $F : \mathbb{H} \to S$ is a homeomorphism.

(Hint: Using the argument principle, the image of $\mathbb{H}$ under F is determined by the image of the real axis and the behavior of F near infinity. Hence, the key step is to determine $\arg(F'(z))$ on the various subintervals of the real line over which it is defined, as well as the existence of a limit of $F(z)$ as $|z| \to \infty$.)

8. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?