Fri 9/3/21, 10:30-11:45 – Math 55a Lecture 2 ①

Announcements: Monday is a holiday ; course feedback survey

* Set theory interlude:

Recall: a map of sets $f: S \to T$ is

- injective if $\forall a,b \in S$, $f(a)=f(b) \Rightarrow a=b$. (or: $a \neq b \Rightarrow f(a) \neq f(b)$). Write $f: S \hookrightarrow T$
- surjective if $\forall c \in T$ $\exists a \in S$ st. $f(a)=c$. Write $f: S \twoheadrightarrow T$.
- a bijection $f: S \xrightarrow{\sim} T$ if both hold.

* Say two sets S, T have the same cardinality if $\exists$ bijection $f: S \to T$, and write $|S|=|T|$. If there exists an injection $f: S \hookrightarrow T$ then write $|S| \leq |T|$. This notation is legit thanks to the Schröder-Benstein theorem:

> If there exist injective maps $f: S \hookrightarrow T$ and $g: T \hookrightarrow S$ then $|S| = |T|$.

(see Halmos Naive set theory p.88 for a proof; build a bijection $S \xrightarrow{\sim} T$ by using f on a subset of S and g^{-1} on the rest).

Ex: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ all have the same cardinality, these are called countably infinite

eg. construct a bijection $\mathbb{N} \to \mathbb{Z}$ by setting $f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if } n \text{ odd.} \end{cases}$

for $\mathbb{Q}$, first understand how to enumerate $\mathbb{N} \times \mathbb{N}$ = pairs of integers.

* On the other hand, $\mathbb{R}$ is uncountable, using Cantor's diagonal argument:
No map $f: \mathbb{N} \to \mathbb{R}$ can be surjective, because:

write decimal or binary expansion of

$$\begin{aligned} f(0) &= a_{00} . a_{01}\, a_{02}\, a_{03} \cdots\cdots \\ f(1) &= a_{10} . a_{11}\, a_{12}\, a_{13} \cdots\cdots \\ f(2) &= a_{20} . a_{21}\, a_{22}\, a_{23} \cdots \\ f(3) &= a_{30} . a_{31}\, a_{32}\, a_{33} \cdots\cdots \end{aligned}$$

then let $y = b_0 . b_1 b_2 b_3 \cdots$ where we choose $b_j \neq a_{jj}$ for each j.
Looking at the j^{th} digit, $y \neq f(j)$ for all $j \in \mathbb{N}$, so f can't be surjective.

* The same argument shows there are arbitrarily large cardinals:
given a set S, let $\mathcal{P}(S) = \{\text{subsets of } S\}$ ("power set of S")

$$\begin{tikzcd} \mathcal{P}(S) \arrow[d, bend left, "A \mapsto \left(\mathbb{1}_A : x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}\right)"] \\ \{0,1\}^S = \{\text{maps } f: S \to \{0,1\}\} \arrow[u, "\simeq"] \arrow[u, bend left, "f \mapsto f^{-1}(1)"'] \end{tikzcd}$$

If S is finite, $|S| = n$, then $|\mathcal{P}(S)| = 2^n$. What if S is infinite?

Thm: if S is infinite then $|\mathcal{P}(S)| > |S|$.

This is just the diagonal argument again. Do you see how?

Pf: (Cantor): given $f: S \to \mathcal{P}(S)$, let $A = \{x \in S \mid x \notin f(x)\}$. Assume $A = f(a)$ for some $a \in S$. Then $a \in A$ iff $a \notin f(a) = A$, contradiction. So $A \notin f(S)$, $\nexists$ surjection. □

Back to groups: **Recall:**

Def^n: A group G = a set with an operation $G\times G \to G$, $(a,b) \mapsto a.b$ such that

(1) identity: $\exists e\in G$ st. $\forall a \in G$, $ae = ea = a$.

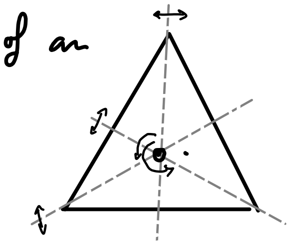
(2) inverse: $\forall a\in G$ $\exists b (=a^{-1}) \in G$ st. $ab = ba = e$.

(3) associativity: $\forall a,b,c \in G$, $(ab)c = a(bc)$.

Examples cont'd: permutations: $\mathrm{Perm}(A) = \{f: A\to A \text{ bijection}\}$, composition $f\circ g$.

The symmetric group on n elements: $S_n = \mathrm{Perm}(\{1,\dots,n\})$

- S_3 has a geometric interpretation if we think of symmetries of an equilateral triangle = rotations which preserve it (3 incl. identity) and reflections (3 of those).



Symmetries permute the vertices, and every permutation of the set of vertices arises from exactly one symmetry (+ composition laws agree).

→ (Other groups arise from symmetries of other geometric figures in $\mathbb{R}^2$ and $\mathbb{R}^3$).

Ex: Groups of matrices: $GL_n(\mathbb{R}) = \{\text{invertible } n\times n \text{ matrices with real coefficients}\}$ "general linear group" (with matrix multiplication)

also $SL_n(\mathbb{R}) = \{\, n\times n \text{ real matrices with determinant } 1\}$ "special linear group".

also $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$ ... or with any field of coeffts $(\mathbb{Q}, \mathbb{Z}/p, \dots)$

Products:

- Given two groups G, H, the product group is $G\times H = \{(g,h) \mid g\in G, h\in H\}$ with composition law $(g,h)\cdot(g',h') = (gg', hh')$.
- If G, H are finite, of order $m = |G|$ and $n = |H|$, then $G\times H$ is a finite group of order mn.
- Similarly for product of n groups:

  Ex: $\mathbb{Z}^n = \{(a_1,\dots,a_n) \mid a_i \in \mathbb{Z}\}$, $(a_1,\dots,a_n) + (b_1,\dots,b_n) = (a_1+b_1, \dots, a_n+b_n)$

  (similarly $\mathbb{Q}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$ with componentwise addition)
- Given infinitely many groups $G_1, G_2, G_3, \dots$, there are two different notions:

  → the direct product $\prod_{i=1}^{\infty} G_i = \{(a_1, a_2, a_3, \dots) \mid a_i \in G_i\}$

  → the direct sum $\bigoplus_{i=1}^{\infty} G_i = \{(a_1,a_2,a_3,\dots) \mid a_i \in G_i, \text{ all but finitely many are identity}\}$

Ex: consider $G_0 = G_1 = \dots = (\mathbb{R}, +)$, denote $(a_0, a_1, a_2, \dots)$ by $\sum a_i x^i$.

Then $\prod_{i=0}^{\infty} \mathbb{R} = \mathbb{R}[[x]]$ formal power series $\sum_{i=0}^{\infty} a_i x^i$ (w/ addition)

$\bigoplus_{i=0}^{\infty} \mathbb{R} = \mathbb{R}[x]$ polynomials $\sum_{\text{finite}} a_i x^i$.

★ Subgroups:

non-empty!

Def: A subgroup H of a group G is a non-empty subset $H\subset G$ which is closed under composition ($a,b\in H \Rightarrow ab\in H$) and inversion ($a\in H \Rightarrow a^{-1}\in H$). These conditions imply $e\in H$. So H (with same operation) is also a group.

Say H is a proper subgroup if $H \subsetneq G$

Examples:
- $(\mathbb{Z},+)\subset(\mathbb{Q},+)\subset(\mathbb{R},+)\subset(\mathbb{C},+)$
- $(\mathbb{Q}^*,\times)\subset(\mathbb{R}^*,\times)\subset(\mathbb{C}^*,\times)\supset(S^1,\times)$
- $\{e\}\subset G$ trivial subgroup
- $H_i\subset G_i \Rightarrow H_1\times\dots\times H_n\subset G_1\times\dots\times G_n$
- $\bigoplus G_i\subset\prod G_i$

Subgroups of $\mathbb{Z}$: given $a\in\mathbb{Z}_{>0}$, $\mathbb{Z}a=\{na \mid n\in\mathbb{Z}\}\subset\mathbb{Z}$ is a subgroup

Prop: All nontrivial subgroups of $(\mathbb{Z},+)$ are of this form.

Proof: This follows from the Euclidean algorithm. Given a nontrivial subgroup $\{0\}\neq H\subset\mathbb{Z}$, there exists $a\in H$ such that $a>0$. Let a_0 be the smallest positive element of H. given any $b\in H$, $b=qa_0+r$ for some $q\in\mathbb{Z}$ and $0\le r<a_0$ (remainder). Since $b\in H$ and $qa_0\in H$, $r\in H$. Since $r<a_0$, by def. of a_0, r must be zero. Hence $b\in\mathbb{Z}a_0$; so $H\subset\mathbb{Z}a_0$, and conversely $\mathbb{Z}a_0\subset H$, so $H=\mathbb{Z}a_0$. □

So. every subgroup of $\mathbb{Z}$ is generated by a single element a_0, in the following sense.

Fact: if $H,H'\subset G$ are two subgroups, then $H\cap H'$ is also a subgroup.

Pf:
- $e\in H\cap H'$ so nonempty
- if $a,b\in H\cap H'$ then $ab\in H$ and $ab\in H'$, so $ab\in H\cap H'$.
- likewise for inverses. □

Similarly for more than two subgroups.

Now: given a subset $S\subset G$ (nonempty), what is the smallest subgroup of G which contains S? This is denoted $\langle S\rangle$ and called the subgroup generated by S.

Answer: look at all subgroups of G which contain S (there's at least G itself!) and take their intersection: $\langle S\rangle = \bigcap\limits_{\substack{S\subset H\subset G\\ \text{subgroup}}} H$.

More useful answer: $\langle S\rangle$ must contain all products of elements of S and their inverses, and these form a subgroup of G, so $\langle S\rangle=\{a_1\dots a_k \mid a_i\in S\cup S^{-1}\ \forall 1\le i\le k\}$

Def: A group is cyclic if it is generated by a single element.

(ex. $\mathbb{Z}$, $\mathbb{Z}/n$. These are in fact the only cyclic groups up to isomorphism).

Ex: $SL_2(\mathbb{Z})=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \,\middle|\, a,b,c,d\in\mathbb{Z} \text{ and } ad-bc=1\right\}$ can be generated by two elements! [exercise! fairly hard without hint].

Homomorphisms:

Def: Given two groups G, H, a homomorphism $\varphi : G \to H$ is a map which respects the composition law: $\forall a, b \in G,\ \varphi(ab) = \varphi(a)\varphi(b)$.
(This implies $\varphi(e_G) = e_H$, and $\varphi(a^{-1}) = \varphi(a)^{-1}$).

Rmk: A pedantic way to state $\varphi(ab) = \varphi(a)\varphi(b)$ is by a commutative diagram

$$\begin{tikzcd} G \times G \arrow[r, "\varphi \times \varphi"'] \arrow[d, "m_G"'] & H \times H \arrow[d, "m_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

"Commutative diagram" means

$$\begin{tikzcd} G \times G \arrow[d, bend left] \arrow[d, bend right] \\ H \end{tikzcd}$$

give the same map: it doesn't matter if we multiply first or apply φ first.

- an isomorphism is a bijective homomorphism (two isomorphic groups are "secretly the same")
- an automorphism is an isomorphism $G \to G$.

Examples: (isomorphisms)

- all groups of order 2 are isomorphic! $S_2 = (\{id, (1\,2)\}, \circ) \cong (\{\pm 1\}, \times) \cong (\mathbb{Z}/_2, +)$ because the table is always

m	e	x
e	e	x
x	x	e

- $(\mathbb{R}, +) \xrightarrow[\exp]{\sim} (\mathbb{R}_+, \times)$ $\quad (\mathbb{R}/\mathbb{Z}, +) \xrightarrow[\exp(2\pi i t)]{\sim} (S^1, \times)$
- $S_3 \cong$ symmetries of △ (permutation of vertices).

Examples: (homomorphisms)

- $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/_n$, $a \mapsto a \bmod n$ (remainder of Euclidean division by n).
- if $n \mid m$, $\mathbb{Z}/_m \twoheadrightarrow \mathbb{Z}/_n$ similarly (eg. $\mathbb{Z}/_{100} \to \mathbb{Z}/_{10}$)
  last 2 digits $\quad$ last digit
- determinant: $GL_n(\mathbb{R}) \to (\mathbb{R}^*, \times)$
  $(\det(AB) = \det(A)\det(B))$.

Definition + Prop^n:

- The kernel of a group homomorphism $\varphi : G \to H$ is
  $$\mathrm{Ker}(\varphi) = \{ a \in G \mid \varphi(a) = e_H \}.$$
- This is a subgroup of G. (check it contains e_G, products, inverses)
- φ is injective iff $\mathrm{Ker}(\varphi) = \{e_G\}$. (using: $\varphi(a) = \varphi(b) \Leftrightarrow a^{-1}b \in \mathrm{Ker}\,\varphi$).

Definition:

- The image of a group homomorphism $\varphi : G \to H$ is
  $$\mathrm{Im}(\varphi) = \varphi(G) = \{ b \in H \mid \exists a \in G \text{ st. } \varphi(a) = b \}$$
- This is a subgroup of H. $\quad \varphi$ is surjective iff $\mathrm{Im}(\varphi) = H$.

Remark: if φ is injective, then G is isomorphic to the subgroup $\mathrm{Im}(\varphi) \subset H$.
(the isomorphism is given by the map $G \to \mathrm{Im}(\varphi)$, $a \mapsto \varphi(a)$).

Example: Let $a \in G$ be any element in a group G, then the map $\varphi : \mathbb{Z} \to G$, $n \mapsto a^n$ is a homomorphism, with image $\langle a \rangle$ the subgroup generated by a.

Def: the <u>order</u> of $a \in G$ = smallest positive k such that $a^k = e$, if it exists. Else say a has infinite order.

⚠ do not confuse order of $a \in G$ with order of G $(= |G|)$. Though, $\text{order}(a) = |\langle a \rangle|$

If a has infinite order then powers of a are all distinct, $\varphi : n \mapsto a^n$ is injective, and $\langle a \rangle$ is isomorphic to $\mathbb{Z}$. If a has finite order k then $\ker(\varphi) = \mathbb{Z}k$, and $\langle a \rangle = \{a^n \mid n = 0, .., k-1\}$ is isomorphic to $\mathbb{Z}/k$.

(This completes the classification of cyclic groups, by the way).

<u>Example</u>: $\mathbb{Z}/6 \xrightarrow{\sim} \mathbb{Z}/2 \times \mathbb{Z}/3$ (observe: $(1,1) \in \mathbb{Z}/2 \times \mathbb{Z}/3$ has order 6, so generates).

$a \mapsto (a \bmod 2, a \bmod 3)$

Similarly, $\gcd(m,n) = 1 \Rightarrow \mathbb{Z}/m \times \mathbb{Z}/n \simeq \mathbb{Z}/mn$. But $\mathbb{Z}/2 \times \mathbb{Z}/2 \not\simeq \mathbb{Z}/4$

$x + x = 0 \ \forall x$ vs. $1 + 1 \neq 0$.

<u>Proposition</u>: Every finite group G is isomorphic to a subgroup of the symmetric group S_n for some n. (In fact we can take $n = |G|$).

<u>Proof</u>: define a map $\phi : G \longrightarrow \text{Perm}(G)$ = permutations of G (bijections $G \to G$) by $\phi(g) = m_g$, where m_g is left multiplication by g, $m_g : G \to G$, $x \mapsto gx$

(Check: why is m_g a permutation?)

- The fact that ϕ is a homomorphism follows from associativity:

$$\left.\begin{aligned} \phi(gh) &= m_{gh} : x \mapsto (gh)x \\ \phi(g) \circ \phi(h) &= m_g \circ m_h : x \mapsto g(hx) \end{aligned}\right\} \text{same}$$

- If $g \neq g'$ then $m_g(e) = g \neq g' = m_{g'}(e)$, so $\phi(g) \neq \phi(g')$.

  Hence ϕ is injective, and $G \simeq \text{Im}(\phi) \subset \text{Perm}(G) \simeq S_{|G|}$. $\square$

An important question in group theory is the <u>classification of finite groups</u> up to isomorphism. This becomes increasingly difficult as $|G|$ increases. The beginning:

- every group of order 2 is isomorphic to $\mathbb{Z}/2$ (by writing the table of the composition law ...).
- similarly, every group of order 3 is $\simeq \mathbb{Z}/3$.
- for order 4, we know $\mathbb{Z}/4$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$.
  (these are different: every nonzero element of $\mathbb{Z}/2 \times \mathbb{Z}/2$ has order 2, while $\mathbb{Z}/4$ has an element of order 4).

  In fact these are the only two groups of order 4 up to iso.

(Classification completed in the 1980s, taking thousands of pages. We'll learn some of the key tools & concepts in the class, but certainly won't tackle the complete classification!).