Def: Given $v_1, \dots, v_n \in V$ vector space over field k,

- $\mathrm{span}(v_1, \dots, v_n) = \{a_1 v_1 + \dots + a_n v_n \mid a_i \in k\}$ smallest subspace of V containing $v_1, \dots, v_n$
- $v_1, \dots, v_n$ are <u>linearly independent</u> if $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$
- $(v_1, \dots, v_n)$ are a <u>basis</u> if they are linearly independent and span V.

($\Rightarrow$ any element of V can be expressed <u>uniquely</u> as $\sum a_i v_i$ for some $a_i \in k$.)

- Say V is <u>finite-dimensional</u> if $\exists$ finite set that spans V.

$\star$ <u>Lemma:</u> if $\{v_1, \dots, v_m\}$ spans V, then a subset of $\{v_1 \dots v_m\}$ is a basis.

<u>Proof:</u> If the $\{v_i\}$ are linearly independent, they form a basis.
Otherwise, there is some linear relation $\sum a_i v_i = 0$, a_i not all zero.
This can be solved for v_i = a linear combination of the others if $a_i \neq 0$.
$\to$ remove v_i, $\{v_j \mid j \neq i\}$ still spans V.
Continue removing elements until the remaining ones are linearly indep^t $\square$

Thus, every finite-dimensional vector space has a basis.

$\star$ <u>Lemma:</u> If $\{v_1, \dots, v_m\}$ are linearly indep^t, there exists a basis of V which contains $\{v_1 \dots v_m\}$

<u>Proof:</u> Let $\{w_1 \dots, w_r\}$ be a spanning set for V. by induction we enlarge $\{v_1, \dots, v_m\}$ to a basis of $W_j = \mathrm{span}(\{v_1, \dots, v_m, w_1, \dots, w_j\}) \subset V$ for each $j = 0, \dots, r$.
For $j = 0$: $\{v_1, \dots, v_m\}$ basis of W_0.
Assuming $\{v_1, \dots, v_m, w_{i_1} \dots w_{i_k}\}$ is a basis of $W_{j-1} = \mathrm{span}(\{v_1 \dots v_m, w_1, \dots, w_{j-1}\})$,
if $w_j \in W_{j-1}$ then we already have a basis of $W_j = W_{j-1}$.
otherwise, $\{v_1 \dots v_m, w_{i_1}, \dots, w_{i_k}, w_j\}$ are linearly indept. (why?) and span W_j.
This ends with a basis of $W_r = V$ (since $\{w_1, \dots, w_r\}$ span). $\square$

- <u>Theorem:</u> If $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$ are bases of V, then $m = n$. (same # elements).

<u>Proof:</u> • We claim $\exists j \in \{1 \dots n\}$ st. $\{v_1, \dots, v_{m-1}, w_j\}$ is a basis.
Indeed, $\{v_1, \dots, v_{m-1}\}$ are linearly independent, but don't span V
(else $v_m \in \mathrm{span}\{v_1 \dots v_{m-1}\}$ gives a linear relation $\sum_{i=1}^{m-1} a_i v_i - v_{m+1} = 0$).
So $\exists j$ st. $w_j \notin \mathrm{span}\{v_1 \dots v_{m-1}\}$ (else $w_1 \dots w_n$ can't span all V).
Now $\{v_1, \dots, v_{m-1}, w_j\}$ are linearly independent (why?),
but using all the v's, can write $w_j = \sum_{i=1}^{m} a_i v_i$ (necess. $a_m \neq 0$)
So $v_m = \frac{1}{a_m}\left(w_j - \sum_{i=1}^{m-1} a_i v_i\right) \in \mathrm{span}(\{v_1 \dots v_{m-1}, w_j\})$

and this implies $\{v_1 \dots v_{m-1}, w_j\}$ span V hence are a basis.

- Repeat this process to exchange one v for one w each time (we don't use the same w twice since the new w we pick has to be independent of the rest of our basis)

We end up with only w's & get an m-element subset of $\{w_1, \dots, w_n\}$ that is also a basis. Necessarily this is all of $\{w_1 \dots w_n\}$, and $m = n$. $\square$

- **Def:** The <u>dimension</u> of V is the cardinality of any basis.

* Given a basis $(v_1, \dots, v_n)$ of V, we get a linear map $\varphi : k^n \longrightarrow V$

$$(a_1, \dots, a_n) \longmapsto \sum a_i v_i$$

Linear independence $\longleftrightarrow$ φ injective

spanning V $\longleftrightarrow$ φ surjective , so φ is an isomorphism!

Every finite-dim. vector space $/k$ is isomorphic to k^n for $n = \dim V$.

(+ basis gives a specific <u>choice</u> of such an isomorphism).

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* Given bases $(v_1 \dots v_n)$ of V and $(w_1 \dots w_m)$ of W, we can represent a linear map $\varphi \in \mathrm{Hom}(V, W)$ by an $m \times n$ <u>matrix</u> $A \in \mathcal{M}_{m,n}$. This amounts to:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle \text{basis}\ \simeq}\uparrow & & \uparrow{\scriptstyle \simeq\ \text{basis}} \\ k^n & \xrightarrow{A} & k^m \end{array} \quad (*)$$

Write $A = (a_{ij})$ $\begin{array}{l} 1 \le i \le m \text{ rows} \\ 1 \le j \le n \text{ columns} \end{array}$ $= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & & \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$

$A : k^n \to k^m$ by multiplication w/ column vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

<u>Notation</u>: $A = \mathcal{M}(\varphi, (v), (w))$ <u>the matrix of φ</u> in given bases

* The entries of A are characterized by: $\displaystyle \varphi(v_j) = \sum_{i=1}^m a_{ij} w_i$.

Ie: the <u>columns</u> of A give the components of $\varphi(v_1), \dots, \varphi(v_n)$ in the basis $\{w_1, \dots, w_m\}$.

Representing any element $x \in V$ as $\displaystyle x = \sum_{i=1}^n x_i v_i \longleftrightarrow$ column vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

and similarly for $y = \varphi(x) \in W$, $y = \sum y_i w_i \Leftrightarrow Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = AX$.

(to be very explicit: $\displaystyle \varphi\Big(\sum_j x_j v_j\Big) = \sum_j x_j \varphi(v_j) = \sum_{i,j} x_j a_{ij} w_i = \sum_i \Big(\sum_j a_{ij} x_j\Big) w_i$.)

* As a memory aid, the isom. $k^n \xrightarrow{\sim} V$ given by the basis can be written symbolically as multiplication of row & column vectors $\underbrace{(v_1 \dots v_n)}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum x_i v_i$.

⚠ these aren't numbers!!

$\varphi\big((v_1 \dots v_n) X\big) = (w_1 \dots w_n) AX$. (compare $(*)$ above)

* This construction gives an isomorphism between the vector spaces $\mathrm{Hom}(V,W)$ and $\mathcal{M}_{m,n}$! In particular $\dim \mathrm{Hom}(V,W) = \dim \mathcal{M}_{m,n} = mn$.
linear maps $\longleftrightarrow$ matrices

- <u>Direct sums and products of vector spaces</u>

Given vector spaces V and W, $V \oplus W = V \times W = \{(v,w) \mid v \in V, w \in W\}$
(with componentwise operations).

Similarly given n vector spaces, $V_1 \oplus \dots \oplus V_n = V_1 \times \dots \times V_n = \{(v_1, \dots v_n) \mid v_i \in V_i\}$
But for infinite collection $(V_i)_{i \in I}$, we have two different constructions:

$\bigoplus_{i \in I} V_i = \{(v_i)_{i \in I} \mid v_i \in V_i$, only finitely many $v_i \neq 0\}$ vs. $\prod_{i \in I} V_i = \{(v_i)_{i \in I} \mid v_i \in V_i\}$

<u>Ex:</u> $\bigoplus_{n \in \mathbb{N}} k \simeq k[x]$ vs. $\prod_{n \in \mathbb{N}} k \simeq k[[x]]$ formal power series.

returning to finite case...

- <u>Sums and direct sums of subspaces:</u>

<u>Def:</u> Given subspaces $W_1, \dots, W_n \subset V$ of some vector space V,
- the <u>span</u> of $W_1, \dots, W_n$ is $W_1 + \dots + W_n = \{w_1 + \dots + w_n \mid w_i \in W_i\} \subset V$.
  Say the W_i span V if $W_1 + \dots + W_n = V$.
- the W_i are <u>independent</u> if $w_1 + \dots + w_n = 0$, $w_i \in W_i \Rightarrow w_i = 0 \ \forall i$.
- if the W_i are independent and span V, say we have a <u>direct sum</u> decomposition $V = W_1 \oplus \dots \oplus W_n$.

* Relation to the previous notion: $\forall i$ we have an inclusion map $W_i \hookrightarrow V$.

These assemble into a linear map
$$\varphi : \bigoplus W_i \longrightarrow V$$
$$(w_1, \dots, w_n) \longmapsto \sum w_i .$$

$W_1 \dots W_n$ span $V \iff \varphi$ surjective, independent $\iff \varphi$ injective.
If both hold, then φ is an isomorphism $\bigoplus W_i \xrightarrow{\sim} V$ and we have a direct sum decomposition.

In this case $\dim(V) = \sum \dim(W_i)$
(get a basis of V by taking the union of bases of $W_1, \dots, W_n$).

★ Case of two subspaces:

$w_1 + w_2 = 0$ iff $w_1 = -w_2 \in W_1 \cap W_2$

we'll see this soon

- W_1, W_2 are independent iff $W_1 \cap W_2 = \{0\}$.
- $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.
- $V = W_1 \oplus W_2$ iff $W_1 \cap W_2 = 0$ and $\dim W_1 + \dim W_2 = \dim W$.

★ Also note: given a subspace $W \subset V$, there exists another subspace W' st $W \oplus W' = V$. (W' is definitely not unique!). To find W': take a basis $\{w_1 \ldots w_r\}$ of W, complete it to a basis $\{w_1, .., w_r, w_1' \ldots w_s'\}$ of V, let $W' = \mathrm{span}(w_1' \cdots w_s')$.

★ Rank and the dimension formula:

given finite-dim. vector spaces V and W, and a linear map $\varphi : V \to W$,

- $\mathrm{Ker}(\varphi) = \{v \in V \,/\, \varphi(v) = 0\} \subset V$
- $\mathrm{Im}(\varphi) = \{w \in W \,/\, \exists v \in V \text{ st. } \varphi(v) = w\} \subset W$ are subspaces of V & W.
- $\dim(\mathrm{Im}\,\varphi)$ is called the rank of φ

Theorem: $\dim \ker(\varphi) + \dim \mathrm{Im}(\varphi) = \dim V$.

Pf: start by choosing a basis $\{u_1 \ldots u_m\}$ for $\ker\varphi$, and complete it to a basis $\{u_1 \ldots u_m, v_1 \cdots v_r\}$ of V. We claim $(\varphi(v_1), \ldots, \varphi(v_r))$ is a basis of $\mathrm{Im}(\varphi)$. Indeed:

- if $w = \varphi(v) \in \mathrm{Im}\,\varphi$, then write $v = \sum a_i u_i + \sum b_j v_j$ and get $\varphi(v) = \sum b_j \varphi(v_j)$ so $\{\varphi(v_j)\}$ span $\mathrm{Im}(\varphi)$
- if $\sum c_j \varphi(v_j) = 0$ then $\varphi(\sum c_j v_j) = 0$, so $\sum c_j v_j \in \ker(\varphi)$ ie. $\sum c_j v_j = \sum a_i u_i$ for some $a_i \in k$.

  But since $\{u_1 \ldots u_m, v_1 \ldots v_r\}$ are linearly indep, this forces all $c_j = 0$ (and $a_i = 0$). Hence $\varphi(v_j)$ are linearly indep.

So now: since $\{\underbrace{u_1 \cdots u_m}_{\substack{m = \dim\ker\varphi \\ ((u_i) \text{ basis of } \ker\varphi)}}, \underbrace{v_1 \ldots v_r}_{\substack{r = \dim \mathrm{Im}(\varphi) = \mathrm{rank}\,\varphi \\ ((\varphi(v_i)) \text{ are a basis of } \mathrm{Im}\,\varphi)}}\}$ basis of V, $m + r = \dim V$. □

Corollary 1: Given a linear map $\varphi : V \to W$, there exist bases of V and W in which the matrix of φ has the form

$$\begin{array}{cc} & \overbrace{}^{r} \\ {\scriptstyle r = \operatorname{rank}\varphi}\left\{\right. & \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

Proof: take basis of V which is $\{v_1, \dots, v_r, \overbrace{u_1, \dots u_m}^{\text{basis of } \ker\varphi}\}$ as above, and complete $\{\varphi(v_1) \dots \varphi(v_r)\}$ (basis of $\operatorname{Im}\varphi$) to a basis of W. $\square$

Corollary 2: For $W_1, W_2 \subset W$ subspaces, $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Proof: consider the map from $V = W_1 \oplus W_2$ to W,

$$\varphi(w_1, w_2) = w_1 + w_2.$$

Then $\operatorname{Im}(\varphi) = W_1 + W_2$, $\ker(\varphi) = \{(u, -u) \mid u \in W_1 \cap W_2\} \simeq W_1 \cap W_2$

so
$$\dim\ker\varphi + \dim\operatorname{Im}\varphi = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$
$$= \dim(V) = \dim(W_1) + \dim(W_2). \quad \square$$