* HW4 posted soon, due Wed. Sept 29, collaboration & OH questions allowed.
* Midterm will be posted Mon. Sept 27 after my office hours; due Fri Oct 1.
  No collaboration, no sources except lecture notes + Artin + Axler

It's meant as a simple check that you know what's going on - it's not meant to be super challenging (no * problems). It may still take time to complete.

Material: everything up to Lecture 10 (Fri 9/24) ≈ Artin through 4.4 / Axler through ch. 5.

Do not discuss the midterm or ask about its contents until after end of week, including in office hours, even if you've turned it in.

E-mail me for clarification requests about the midterm after it gets posted.

Last time: $\varphi: V \to W$ linear map, if we choose bases $\{v_1 \dots v_n\}$ for V, $\{w_1 \dots w_m\}$ for W, then we can represent φ by a matrix $A = \mathcal{M}(\varphi, \{v_j\}, \{w_i\})$

(j^{th} column of A = components of $\varphi(v_j)$ in basis $\{w_i\}$) so that

if $v \in V$ is rep^d by column vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ (ie. $v = \sum x_j v_j$)

then $\varphi(v) \in W$ is rep^d by $Y = AX = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$, $y_i = \sum_j a_{ij} x_j$ (ie. $\varphi(v) = \sum y_i w_i$).

Think of this as: $\varphi((v_1,\dots,v_n)X) = (w_1,\dots,w_n)AX$.

* Change of basis: What if we chose different basis for V and/or W?

If we change basis from $(v_1 \dots v_n)$ to $(v_1' \dots v_n')$, write $v_j' = \sum_{i=1}^n p_{ij} v_i$

and get an $n \times n$ matrix P whose j^{th} column gives the components of v_j' in the basis $(v_1 \dots v_n)$. Symbolically $(v_1' \dots v_n') = (v_1 \dots v_n) P$.

So: $(v_1' \dots v_n') X' = (v_1 \dots v_n) P X'$ ie. the element of V described by a column vector X' in new basis is described by $X = PX'$ in old basis.

More conceptually: $P = \mathcal{M}(id_V, (v'), (v))$!

Do the same for W, in reverse: let $Q = \mathcal{M}(id_W, (w), (w'))$ ie. $(w_1 \dots w_m) = (w_1' \dots w_m') Q$.

Hence: $\varphi((v_1' \dots v_n')X') = \varphi((v_1 \dots v_n)PX') = (w_1 \dots w_m)APX' = (w_1' \dots w_m') QAPX'$

ie. $\mathcal{M}(\varphi, (v'), (w')) = QAP$.

* In particular, if $V = W$ and change basis, for $\varphi \in Hom(V,V)$,
  $A = \mathcal{M}(\varphi, (v), (v))$ and $A' = \mathcal{M}(\varphi, (v'), (v'))$ are related by $A' = P^{-1}AP$.

$\longrightarrow$ But ... the whole point of linear algebra is to avoid all this and work with linear maps in a coordinate-free language as much as possible.

* **Quotient spaces:** Let V be a vector space over a field k, $U \subset V$ a subspace.

Def: The *quotient space* $V/U = \{v+U\}$ is the space of cosets of U in V, with addition $(v+U)+(w+U) = (v+w)+U$
scalar multiplication $a(v+U) = av+U$.

The linear map $\begin{array}{c} V \xrightarrow{q} V/U \\ v \mapsto v+U \end{array}$ is surjective, with kernel $= U$. Hence, we get an exact sequence $0 \to U \to V \to V/U \to 0$.

By the dimension formula $(\dim \ker q + \dim \operatorname{Im} q = \dim V)$, we have:
$$\dim(V/U) = \dim V - \dim U.$$

Remark: 1) given a linear map $V \xrightarrow{\varphi} W$, if $U \subset \ker\varphi$ (ie. $\varphi_{|U} = 0$)
then φ factors through V/U, ie.

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "q"'] & & W \\ & V/U \arrow[ur, "\bar\varphi"'] & \end{tikzcd}$$

$\exists \bar\varphi : V/U \to W$ st. $\varphi = \bar\varphi \circ q$.

(define $\bar\varphi(v+U) = \varphi(v)$, this is indep^t of choice of v in coset)

Conversely, given $\bar\varphi \in \operatorname{Hom}(V/U, W)$, $\varphi = \bar\varphi \circ q : V \to W$ has $U \subset \ker\varphi$.

Hence: $\{\varphi \in \operatorname{Hom}(V,W) \mid U \subset \ker\varphi\} \cong \operatorname{Hom}(V/U, W)$.

2) there is a bijection $\{\text{subspaces of } V \text{ containing } U\} \longleftrightarrow \{\text{subspaces of } V/U\}$
$$W \subset V, (W \supset U) \longmapsto W/U = \{w+U, w \in W\}$$
Conversely, $$q^{-1}(\overline{W}) \subset V \longleftarrow \overline{W} \subset V/U$$
$(U = q^{-1}(0) \subset q^{-1}(\overline{W})$ since $0 \in \overline{W})$.

Dual spaces: Let V be a vector space over a field k.

Def: The *dual vector space* is the space of *linear functionals* on V, ie. linear maps $V \to k$: $V^* = \operatorname{Hom}(V,k) = \{\text{linear maps } \ell : V \to k\}$

Ex: if $V = k^n = \{(x_1, \dots x_n) \mid x_i \in k\}$, any tuple $(a_1, \dots, a_n)$, $a_i \in k$ determines a map $\ell_a : k^n \to k$, $\ell_a(x_1, \dots, x_n) = \sum a_i x_i$.

Conversely, let e_i = standard basis of k^n, given $\ell : k^n \to k$, let $a_i = \ell(e_i)$,
then $\ell(x_1, \dots, x_n) = \ell(\sum x_i e_i) = \sum a_i x_i$, ie. $\ell = \ell_a$.

So: $(k^n)^* = \{(a_1 \dots a_n) \mid a_i \in k\} \cong k^n$.

* More generally, given a finite dim.l V and a basis $\{e_1 \dots e_n\}$, then any linear map $\ell : V \to k$ is determined by $\ell(e_i)$, so we get an isomorphism $V^* \simeq k^n$, $\ell \mapsto (\ell(e_1), \dots, \ell(e_n))$

Equivalently, we get a basis of V^* consisting of the linear functionals $e_1^*, \dots, e_n^*$ s.t. $e_i^*(e_i) = 1$ and $e_i^*(e_j) = 0$ for $j \neq i$. (then $\ell = \sum_{i=1}^n \ell(e_i) e_i^*$!)

This is called the <u>dual basis</u>!

* However, there is <u>not</u> a natural map $V \to V^*$. Despite the above about bases. Each element of the dual basis e_i^* depends not just on e_i, but on <u>all</u> e_j's. There's no such thing as "the dual of a vector".

* On the other hand, we do have a natural map $V \xrightarrow{ev} (V^*)^*$ ("evaluation")
$$v \longmapsto ev_v : V^* \to k, \quad \ell \mapsto \ell(v)$$

* If V is finite-dimensional, then by working in bases $\{e_1 \dots e_n\}$, dual basis $\{e_1^* \dots e_n^*\}$, & double dual basis $\{e_1^{**}, \dots, e_n^{**}\}$, we see that $e_i^{**}(e_j^*) = e_j^*(e_i)$ ✓ and so $ev(e_i) = e_i^{**}$, hence ev is an isom. Thus:

⇒ <u>Prop</u>: ‖ If V is finite-dimensional then $V \cong V^{**}$, $v \mapsto (\ell \mapsto \ell(v))$ isomorphism.

* When V is infinite-dimensional, $ev : V \to V^{**}$ is injective, but <u>not</u> an isom! The reason is: Assume V has a basis $\{e_i\}_{i \in I}$, so every element of V is uniquely $\sum_{i \in I} x_i e_i$, w/ only finitely many nonzero x_i $(V \simeq \bigoplus_{i \in I} k \cdot e_i)$.

Then $\forall (a_i)_{i \in I} \in \prod_{i \in I} k$, $\ell_a : V \to k$, $\sum x_i e_i \mapsto \sum x_i a_i$ is a well defined element of V^*, characterized by $\ell(e_i) = a_i \ \forall i \in I$.

So: $V^* \simeq \prod_{i \in I} k$, which is larger, and the linear functionals e_i^* $(a_i = 1, a_j = 0 \ \forall j \neq i)$ do not span V^*. (Can complete to a basis via Zorn's lemma.) A similar enlargement happens again when passing from V^* to V^{**}.

<u>Def</u>: ‖ The <u>annihilator</u> of a subspace $U \subset V$ is $\mathrm{Ann}(U) = \{\ell : V \to k \ / \ \ell_{|U} = 0\} \subset V^*$. (This is a subspace of V^*).

- $V^* \to U^*$, $\ell \mapsto \ell_{|U}$ is surjective with kernel $= \mathrm{Ann}(U)$, so $0 \to \mathrm{Ann}(U) \to V^* \to U^* \to 0$, $\ell \mapsto \ell_{|U}$.
This in turn implies $V^*/\mathrm{Ann}(U) \simeq U^*$.

- Also, we've seen above: $\{\ell \in \mathrm{Hom}(V,k) \,/\, U \subset \ker \ell\} \simeq \mathrm{Hom}(V/U, k)$.

  Hence: $\mathrm{Ann}(U) \simeq (V/U)^*$

- Either way, this implies: $\dim \mathrm{Ann}(U) = \dim V - \dim U$.

Def: Given a linear map $\varphi: V \to W$, the transpose of φ, $\varphi^*: W^* \to V^*$ defined as follows: given a linear functional $\ell: W \to k$, composing with $\varphi: V \to W$ gives a linear map $\ell \circ \varphi: V \to k$. Thus, $\varphi^*: W^* \longrightarrow V^*$

$$\ell \mapsto \varphi^*(\ell) := \ell \circ \varphi.$$

(check: φ^* is linear)

Check:
- given a basis (e_i) of V, elements of V are represented by column vectors X

  $V^* = \mathrm{hom}(V,k)$ —"— row vectors Y

  Applying a linear functional $\ell \in V^*$ to a vector $v \in V \Leftrightarrow YX \in k$.

- if $\mathcal{M}(\varphi, (e_i), (f_j)) = A$, then $\mathcal{M}(\varphi^*, (f_j^*), (e_i^*)) = A^T$ transpose matrix

  (This is because: given $\ell \in W^*$ and $v \in V$, $\ell(\varphi(v)) = (\varphi^*(\ell))(v) = YAX$ so φ^*, viewed as operation on row vectors, is $Y \mapsto YA$.
  Meanwhile the dual bases give a description of elements of V^*, W^* by column vectors, which are the transposes of the row vectors. The claim then follows since $\varphi^*\ell$ as column vector is $(YA)^T = A^T Y^T$.)

Prop: (In the finite dim. case) φ is injective iff φ^* is surjective

φ is surjective iff φ^* is injective

follows from: **Prop:** (1) $\ker(\varphi^*) = \mathrm{Ann}(\mathrm{Im}\,\varphi)$

(2) $\mathrm{Im}(\varphi^*) = \mathrm{Ann}(\ker \varphi)$ ← assuming finite dim.

Proof: (1) $\ell \in \mathrm{Ann}(\mathrm{Im}\,\varphi) \Leftrightarrow \ell(\varphi(v)) = 0 \;\forall v \in V \Leftrightarrow \varphi^*(\ell) = \ell \circ \varphi = 0 \Leftrightarrow \ell \in \ker \varphi^*$.

(2) If $\ell' = \varphi^*(\ell) \in \mathrm{Im}(\varphi^*)$ then $\ell' = \ell \circ \varphi$ so $\ell'|_{\ker \varphi} = 0$. So $\mathrm{Im}(\varphi^*) \subset \mathrm{Ann} \ker \varphi$.

Dim. formula and (1) imply $\mathrm{rank}(\varphi^*) = \mathrm{rank}(\varphi)$, hence the inclusion is an equality. □

Linear operators: A linear operator on V (aka endomorphism of V) is a linear map $\varphi: V \to V$.

Notation: $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

* When using a basis to express $\varphi \in \mathrm{Hom}(V,V)$ by a (square) matrix, we want to use the same basis on each side: $A = \mathcal{M}(\varphi, (e_i), (e_i))$, transforms by $P^{-1}AP$.

* New thing: we can compose linear operators with each other $\varphi\psi = \varphi \circ \psi : V \to V$ or with themselves; $\varphi^n = \underbrace{\varphi \circ \dots \circ \varphi}_{n \text{ times}}$, or even apply polynomials:
$p = \sum a_n x^n \rightsquigarrow p(\varphi) = \sum a_n \varphi^n : V \to V$.

Hom(V,V) is a (noncommutative) ring.

* given vector spaces V_1, V_2 and linear operators $\varphi_i : V_i \to V_i$, we can define
$\varphi = \varphi_1 \oplus \varphi_2 : V_1 \oplus V_2 \to V_1 \oplus V_2$ operator on $V = V_1 \oplus V_2$.

The operator φ leaves the subspaces $V_1, V_2 \subset V$ invariant: $\varphi(V_i) \subset V_i$, and working in a basis of V st. $e_1 \dots e_m \in V_1$, $e_{m+1} \dots e_n \in V_2$, the matrix of φ is block diagonal: $\left(\begin{array}{c|c} \varphi_1 & 0 \\ \hline 0 & \varphi_2 \end{array}\right)$. Conversely, if $V = V_1 \oplus V_2$ and $\varphi(V_i) \subset V_i$ then φ is of this form.

More generally, if we only assume $\varphi : V \to V$ and $V_1 \subset V$ is invariant ($\varphi(V_1) \subset V_1$) but not necess. V_2, then the matrix of φ would be block triangular: $\left(\begin{array}{c|c} \varphi_{/V_1} & * \\ \hline 0 & * \end{array}\right)$

So: a typical way to study $\varphi : V \to V$ is to look for invariant subspaces.

* If $U \subset V$ is invariant and $\dim U = 1$ (so: $U = k.v$ for some $v \in V$), then necessarily $\varphi(v) = \lambda v$ for some $\lambda \in k$.

In this case v is called an eigenvector of φ, and λ is called the eigenvalue corresponding to v.

* If we can find a basis of V consisting of eigenvectors of φ, then we have diagonalized φ, ie. found a basis where its matrix is diagonal

$$\varphi(v_i) = \lambda_i v_i \qquad \begin{array}{c} \begin{array}{ccc} v_1 & \dots & v_n \end{array} \\ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \end{array}$$

This is the best outcome, but not always possible!

Ex: $V = \mathbb{R}^2$, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ has eigenvectors $\begin{matrix}(1,0)\\(0,1)\end{matrix}$ (or any multiples) with eigenvalues $\begin{matrix}\lambda\\ \mu\end{matrix}$.

However $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only one eigenvector (up to scaling!) $(1,0)$ with eigenvalue 1, NOT diagonalizable.

Next time, we'll learn more about eigenvectors, invariant subspaces, etc.