Recall: the dual vector space of V is $V^* = \mathrm{Hom}(V, k)$

Def: Given a linear map $\varphi: V \to W$, the transpose of φ, $\varphi^*: W^* \to V^*$ defined as follows: given a linear functional $l: W \to k$, composing with $\varphi: V \to W$ gives a linear map $l \circ \varphi: V \to k$. Thus, $\varphi^*: W^* \longrightarrow V^*$

(check: φ^* is linear) $\qquad l \mapsto \varphi^*(l) := l \circ \varphi$.

Check: • given a basis (e_i) of V, elements of V are represented by column vectors X
$V^* = \mathrm{hom}(V, k)$ —— " —— row vectors Y

Applying a linear functional $l \in V^*$ to a vector $v \in V \Leftrightarrow YX \in k$.

• if $\mathcal{M}(\varphi, (e_i), (f_j)) = A$, then $\mathcal{M}(\varphi^*, (f_j^*), (e_i^*)) = A^T$ transpose matrix

(This is because: given $l \in W^*$ and $v \in V$, $l(\varphi(v)) = (\varphi^*(l))(v) = YAX$
so φ^*, viewed as operation on row vectors, is $Y \mapsto YA$.
Meanwhile the dual bases give a description of elements of V^*, W^* by column vectors, which are the transposes of the row vectors. The claim then follows since $\varphi^* l$ as column vector is $(YA)^T = A^T Y^T$.)

Prop: (In the finite dim. case) φ is injective iff φ^* is surjective
φ is surjective iff φ^* is injective

follows from: **Prop:**
(1) $\ker(\varphi^*) = \mathrm{Ann}(\mathrm{Im}\,\varphi)$ $\quad (= \{l \in W^* \,/\, l_{|\mathrm{Im}\,\varphi} = 0\})$.
(2) $\mathrm{Im}(\varphi^*) = \mathrm{Ann}(\ker \varphi)$ ← assuming finite dim.

Proof: (1) $l \in \mathrm{Ann}(\mathrm{Im}\,\varphi) \Leftrightarrow l(\varphi(v)) = 0\ \forall v \in V \Leftrightarrow \varphi^*(l) = l \circ \varphi = 0 \Leftrightarrow l \in \ker \varphi^*$.
(2) If $l' = \varphi^*(l) \in \mathrm{Im}(\varphi^*)$ then $l' = l \circ \varphi$ so $l'_{|\ker \varphi} = 0$. So $\mathrm{Im}(\varphi^*) \subset \mathrm{Ann}\ker\varphi$.
Dim. formula and (1) imply $\mathrm{rank}(\varphi^*) = \mathrm{rank}(\varphi)$, hence the inclusion is an equality. □

Linear operators: A linear operator on V (aka endomorphism of V) is a linear map $\varphi: V \to V$.

Notation: $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

* When using a basis to express $\varphi \in \mathrm{Hom}(V, V)$ by a (square) matrix, we want to use the same basis on each side: $A = \mathcal{M}(\varphi, (e_i), (e_i))$, transforms by $P^{-1}AP$.
* New thing: we can compose linear operators with each other $\varphi\psi = \varphi \circ \psi: V \to V$ or with themselves, $\varphi^n = \varphi \circ \dots \circ \varphi$, or even apply polynomials:

$\Rightarrow \mathrm{Hom}(V, V)$ is a (noncommutative) ring. $\qquad p = \sum a_n x^n \rightsquigarrow p(\varphi) = \sum a_n \varphi^n : V \to V$.

* given vector spaces V_1, V_2 and linear operators $\varphi_i : V_i \to V_i$, we can define
$\varphi = \varphi_1 \oplus \varphi_2 : V_1 \oplus V_2 \to V_1 \oplus V_2$ operator on $V = V_1 \oplus V_2$.
The operator φ leaves the subspaces $V_1, V_2 \subset V$ <u>invariant</u> : $\varphi(V_i) \subset V_i$,
and working in a basis of V st. $e_1 \dots e_m \in V_1$, $e_{m+1} \dots e_n \in V_2$, the matrix
of φ is <u>block diagonal</u> : $\left(\begin{array}{c|c} \varphi_1 & 0 \\ \hline 0 & \varphi_2 \end{array}\right)$. Conversely, if $V = V_1 \oplus V_2$ and
$\varphi(V_i) \subset V_i$ then φ is of this form.
More generally, if we only assume $\varphi : V \to V$ and $V_1 \subset V$ is invariant
$(\varphi(V_1) \subset V_1)$ but not necess. V_2, then the matrix of φ would be
<u>block triangular</u> : $\left(\begin{array}{c|c} \varphi_{/V_1} & * \\ \hline 0 & * \end{array}\right)$

<u>So</u>: a typical way to study $\varphi : V \to V$ is to look for <u>invariant subspaces</u>.

* If $U \subset V$ is invariant and $\dim U = 1$ (so : $U = k.v$ for some $v \in V$),
then necessarily $\varphi(v) = \lambda v$ for some $\lambda \in k$.

<u>Def</u>: An <u>eigenvector</u> of $\varphi : V \to V$ is a vector $v \in V, v \neq 0$, st. $\varphi(v) = \lambda v$ for some $\lambda \in k$. λ is called the <u>eigenvalue</u> corresponding to v.

* If we can find a basis of V consisting of eigenvectors of φ, then we have
<u>diagonalized</u> φ, ie. found a basis where its matrix is <u>diagonal</u>

$$\varphi(v_i) = \lambda_i v_i \qquad \begin{array}{c} v_1 \ \cdots \ v_n \\ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \end{array}$$

This is the best outcome, but not always possible!

<u>Ex</u>: $V = \mathbb{R}^2$, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ has eigenvectors $\begin{matrix}(1,0) \\ (0,1)\end{matrix}$ (or any multiples) with eigenvalues $\begin{matrix}\lambda \\ \mu\end{matrix}$.

However $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only one eigenvector $(1,0)$ with eigenvalue 1,
(up to scaling!) <u>NOT</u> diagonalizable.

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ doesn't even have any eigenvectors (see HW4 Problem 1).

<u>Prop</u>: Eigenvectors of $\varphi : V \to V$ with distinct eigenvalues are linearly independent.

<u>Pf</u>: Assume $v_1 \dots v_\ell$ are eigenvectors with $\varphi(v_i) = \lambda_i v_i$, λ_i all distinct.
Assume there is a linear relation $a_1 v_1 + \dots + a_\ell v_\ell = 0$ with a_i not all zero.

and this has the fewest (≥ 2) possible nonzero a_i of any such relation

Then $\varphi(\sum a_i v_i) = \sum a_i \varphi(v_i) = \sum a_i \lambda_i v_i = 0$ another linear relation!

$$a_1 \lambda_1 v_1 + \dots + a_\ell \lambda_\ell v_\ell = 0.$$

Fix i st. $a_i \neq 0$, and subtract:

$$a_1(\lambda_1 - \lambda_i) v_1 + \dots + a_\ell(\lambda_\ell - \lambda_i) v_\ell = 0$$

$\to$ linear relation where coefficient of v_i is now zero, but all other nonzero coefficients (at least one) remain nonzero (since $\lambda_j - \lambda_i \neq 0$) - contradicts minimality assumption. $\square$

<u>Corollary</u>: The number of distinct eigenvalues of $\varphi \in \mathrm{Hom}(V,V)$ is at most $n = \dim V$, and if equality holds then φ is diagonalizable.

<u>Digression</u>: <u>Def</u>: A field k is <u>algebraically closed</u> if every nonconstant polynomial $p \in k[x]$ has a root in k, ie. $\exists \alpha \in k$ st. $p(\alpha) = 0$.

If so, then by division algorithm for polynomials, can write $p = (x-\alpha) q$, and repeating, we get $p = c(x-\alpha_1)\dots(x-\alpha_d)$. ($d = \deg p$, $\alpha_i \in k$).

* <u>Fundamental theorem of algebra</u>: $\mathbb{C}$ is algebraically closed. (proof is <u>not</u> pure algebra; we'll discuss it in Math 556).
* If k is not algebraically closed then there exists an alg. closed field $\bar{k} \supset k$, constructed from k by adjoining roots of polynomials $\in k[x]$.

Eg. $\overline{\mathbb{R}} = \mathbb{C}$, whereas $\overline{\mathbb{Q}} = \{$all roots of polynomial eq^{ns} with $\mathbb{Q}$-coeffts$\} \subset \mathbb{C}$

(<u>fact</u>: polynomials in $\overline{\mathbb{Q}}[x]$ have roots in $\overline{\mathbb{Q}}$)

<u>Prop</u>: If k is algebraically closed, V a finite dimensional vector space over k, then any linear operator $\varphi: V \to V$ has an eigenvector, ie. $\exists v \in V - \{0\}$, $\exists \lambda \in k$ st. $\varphi(v) = \lambda v$.

<u>Proof</u>: Let $n = \dim V$, and take any nonzero vector $v \in V$. Then $\underbrace{v, \varphi(v), \dots, \varphi^n(v)}_{n+1 \text{ vectors}}$ must be linearly dependent.

So $\exists a_0, \dots, a_n \in k$ (not all zero) st. $a_0 v + a_1 \varphi(v) + \dots + a_n \varphi^n(v) = 0$.

Since k is algebraically closed, we can factor the polynomial $\sum a_i x^i$, hence $a_0 + a_1\varphi + \dots + a_n \varphi^n = c(\varphi - \lambda_1)\dots(\varphi - \lambda_d)$, $c \neq 0$, $\lambda_i \in k$.

(!! the product here is composition of operators, but this is legit !!)

Now, $(\varphi-\lambda_1)\cdots(\varphi-\lambda_d) : V \to V$ has a nontrivial kernel $(\ni v)$, which implies that at least one of $\varphi-\lambda_i$ is not an isomorphism, hence $\exists\, i \in \{1..d\}$ and $w \in V-\{0\}$ s.t. $w \in \mathrm{Ker}(\varphi-\lambda_i)$, i.e. $\varphi(w)=\lambda_i w$. □

Corollary: Given $\varphi: V \to V$ over an algebraically closed field k, there exists a basis $(v_1,\dots,v_n)$ of V in which the matrix of φ is upper-triangular. $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

(i.e. each subspace $V_k = \mathrm{span}(v_1 \dots v_k) \subset V$ is invariant)

Proof: Induction on $\dim V$: If $\dim V = 1$, then any nonzero vector v_1 gets mapped to a multiple of itself ✓. (any 1×1 matrix is triangular)

- Assume result true for dim. $\leq n-1$, and consider $\varphi: V \to V$ with $\dim V = n$. By Lemma, φ has at least one eigenvalue $\lambda \in k$. Let $U = \mathrm{Im}(\varphi-\lambda)$. Since $\varphi-\lambda$ has nontrivial kernel (= eigenvectors for λ), $\dim U < \dim V$. Moreover, we claim U is an invariant subspace for φ.

  Indeed: if $u = (\varphi-\lambda)v \in \mathrm{Im}(\varphi-\lambda) = U$, then
  $$\varphi(u) = \varphi(\varphi-\lambda)v = (\varphi-\lambda)\varphi(v) \in \mathrm{Im}(\varphi-\lambda) = U.$$

  Now, by induction, $\varphi_{|U} \in \mathrm{Hom}(U,U) \rightsquigarrow \exists$ basis $u_1,\dots,u_m$ of U in which $\varphi_{|U}$ is upper-triangular. $(\varphi(u_i) \in \mathrm{span}(u_1 \dots u_i))$

  Complete to a basis $(u_1 \dots u_m, v_1 \dots v_k)$ of V. Then:
  - $\varphi(u_i) \in \mathrm{span}(u_1 \dots u_i)$ ✓
  - $\varphi(v_i) = \underbrace{(\varphi-\lambda)v_i}_{\in U} + \lambda v_i \in \mathrm{span}(u_1, \dots u_m, v_i)$. ✓

  $$\Rightarrow M(\varphi) = \begin{array}{cc} \overbrace{\qquad}^{U} & \\ \left(\begin{array}{c|c} M(\varphi_{|U}) & * \\ \hline 0 & \begin{matrix} \lambda & 0 \\ 0 & \ddots\ \lambda \end{matrix} \end{array}\right) & \end{array}$$

□

* **Rmk:** there's another proof that is easier to discover but harder to follow: again by induction, but now start from $V_0 = k \cdot v_0$ where v_0 is an eigenvector of φ, and let $U = V/V_0$, $q: V \to U$ quotient.

Using $\varphi(V_0) \subset V_0$, $\exists\, \bar\varphi : U \to U$ st.
$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "q"'] & V \arrow[d, "q"] \\ U \arrow[r, "\bar\varphi"] & U \end{tikzcd}$$
commutes

(because: $(q\circ\varphi)_{|V_0} = 0$ so $q\circ\varphi : V \to U$ factors through $V/V_0 = U$).

By induction hypothesis, $\exists$ basis $u_1,\dots u_{n-1}$ of U st. $\bar\varphi(u_i) \in \mathrm{span}(u_1 \dots u_i)$.

Let $v_i \in V$ such that $q(v_i) = u_i$. Then $q(\varphi(v_i)) \in \mathrm{span}(u_1 \dots u_i)$

(Note: $(v_0 \dots v_{n-1})$ basis of V). $\Rightarrow \varphi(v_i) \in \mathrm{span}(v_0, v_1 \dots v_i)$.

$$M(\varphi) = \begin{array}{c} \begin{array}{cc} v_0 & \qquad \end{array} \\ \left(\begin{array}{c|c} \lambda & * \\ \hline 0 & \\ \vdots & M(\bar\varphi) \\ 0 & \end{array}\right) \end{array}$$

□

Now suppose we have $\varphi: V \to V$ and a basis $(v_1, \dots, v_n)$ of V st $\mathcal{M}(\varphi) = A$ is upper-triangular, ie. each $V_i = \mathrm{span}(v_1, \dots, v_i)$ is an invariant subspace of φ.
Denote by $\lambda_i = a_{ii}$ the diagonal entries of A.

<u>Lemma:</u> φ is invertible iff all the diagonal entries of A are nonzero.

<u>Pf:</u>
- if all λ_i are nonzero then φ is surjective (hence isom.) since
  $\varphi(v_1) = \lambda_1 v_1$, $\lambda_1 \neq 0$ so $v_1 \in \mathrm{Im}\,\varphi$
  $\varphi(v_2) = \lambda_2 v_2 + a_{12} v_1$, $\lambda_2 \neq 0$ so $v_2 = \frac{1}{\lambda_2}(\varphi(v_2) - a_{12} v_1) \in \mathrm{Im}\,\varphi$
  etc. $\Rightarrow v_i \in \mathrm{Im}\,\varphi \ \forall i$.
- if $\lambda_i = 0$ then $\varphi(V_i) \subset V_{i-1}$ so $\varphi_{|V_i}$ has nontrivial kernel (since $\mathrm{rk}\,\varphi_{|V_i} \leq \dim V_{i-1} < \dim V_i$); hence $\ker \varphi \neq 0$, not invertible. □

<u>Corollary:</u> The following are equivalent:
(1) λ is an eigenvalue of φ
(2) $\varphi - \lambda$ is not invertible
(3) $\lambda = \lambda_i$ for some diagonal entry of any upper-triangular matrix A representing φ.

((1)$\Leftrightarrow$(2) since eigenvectors $= \ker(\varphi - \lambda)$, and (2)$\Leftrightarrow$(3) by applying the lemma to $\varphi - \lambda$ and matrix $A - \lambda I$).