Last time, we talked about linear operators $\varphi: V \to V$, their invariant subspaces ($U \subset V$ st. $\varphi(U) \subset U$), and eigenvectors ($v \neq 0$ st. $\varphi(v) = \lambda v$, ie. $v \in \mathrm{Ker}(\varphi - \lambda I)$).

Over any field:

- eigenvectors need not exist; eigenvectors for distinct λ are linearly independent;
- if $\exists\ n = \dim V$ distinct eigenvalues then φ is diagonalizable: $\exists$ basis st. $\mathcal{M}(\varphi) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

We saw:

<u>Prop</u>: If k is algebraically closed (eg. $\mathbb{C}$), V a finite dimensional vector space over k, then any linear operator $\varphi: V \to V$ has an eigenvector.

(Idea: given any nonzero $v \in V$, rewrite a linear relation between $v, \varphi(v), \ldots, \varphi^n(v)$ as $P(\varphi)(v) = 0$, $P \in k[x] \Rightarrow (\varphi - \lambda_1)\ldots(\varphi - \lambda_d)v = 0 \Rightarrow \exists i$ st. $\mathrm{Ker}(\varphi - \lambda_i) \neq 0$.).

<u>Corollary</u>: Given $\varphi: V \to V$ over an algebraically closed field k, there exists a basis $(v_1, \ldots, v_n)$ of V in which the matrix of φ is upper-triangular. $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$
(ie. each subspace $V_k = \mathrm{span}(v_1 \ldots v_k) \subset V$ is invariant)

(see Axler thm 5.27 proof 1 for another proof that is slightly more elementary but less intuitive)
We follow proof 2 instead.

<u>Proof</u>: • Induction on $\dim V$: If $\dim V = 1$, then any nonzero vector v_0 gets mapped to a multiple of itself ✓. (any 1×1 matrix is triangular)

• Assume the result is true for dim. n, and consider $\varphi: V \to V$ with $\dim V = n+1$.
Since k is alg. closed, φ has at least one eigenvector $v_0 \in V$, $\varphi(v_0) = \lambda v_0$.
Let $V_0 = \mathrm{span}(v_0)$, and let $U = V/V_0$, $q: V \to U$ quotient map (ie. work "mod v_0").
Since $\varphi(V_0) \subset V_0$, $(q \circ \varphi)_{|V_0} = 0$ so $q \circ \varphi$ factors through $V/V_0 = U$. So:

$\exists\ \bar\varphi: U \to U$ st.
$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "q"'] & V \arrow[d, "q"] \\ U \arrow[r, "\bar\varphi"] & U \end{tikzcd}$$
commutes (intuitively $\bar\varphi$ = "φ mod v_0").

($\dim U = \dim V - \dim V_0 = n$)

By induction hypothesis, $\exists$ basis $u_1 \ldots u_n$ of U st. $\bar\varphi(u_i) \in \mathrm{span}(u_1 \ldots u_i)$
Let $v_i \in V$ such that $q(v_i) = u_i$. (ie. $u_i = v_i + V_0$).

(Note: $v_1 + V_0, \ldots, v_n + V_0$ span V/V_0 $\Rightarrow (v_0, v_1, \ldots, v_n)$ span V hence are a basis).

Then $q(\varphi(v_i)) = \bar\varphi(u_i) \in \mathrm{span}(u_1 \ldots u_i)$
$\Rightarrow \varphi(v_i) \in \mathrm{span}(v_1 + V_0, \ldots, v_i + V_0) = \mathrm{span}(v_0, v_1, \ldots, v_i)$.

(In fact $\mathcal{M}(\varphi) = \begin{array}{c} v_0 \\ \left(\begin{array}{c|c} \lambda & * \\ \hline 0 & \\ \vdots & \mathcal{M}(\bar\varphi) \\ 0 & \end{array}\right) \end{array}$) $\square$

Now suppose we have $\varphi: V \to V$ and a basis $(v_1, .., v_n)$ of V st $\mathcal{M}(\varphi) = A$ is upper-triangular, ie. each $V_i = \mathrm{span}(v_1, .., v_i)$ is an invariant subspace of φ.
Denote by $\lambda_i = a_{ii}$ the diagonal entries of A.

<u>Lemma:</u> φ is invertible iff all the diagonal entries of A are nonzero.

<u>Pf:</u>
- if all λ_i are nonzero then φ is surjective (hence isom.) since
  $\varphi(v_1) = \lambda_1 v_1$, $\lambda_1 \neq 0$ so $v_1 \in \mathrm{Im}\,\varphi$
  $\varphi(v_2) = \lambda_2 v_2 + a_{12} v_1$, $\lambda_2 \neq 0$ so $v_2 = \frac{1}{\lambda_2}(\varphi(v_2) - a_{12} v_1) \in \mathrm{Im}\,\varphi$
  etc. $\Rightarrow v_i \in \mathrm{Im}\,\varphi\ \forall i$.
- if $\lambda_i = 0$ then $\varphi(V_i) \subset V_{i-1}$ so $\varphi_{|V_i}$ has nontrivial kernel (since $\mathrm{rk}\,\varphi_{|V_i} \leq \dim V_{i-1} < \dim V_i$); hence $\ker\varphi \neq 0$, not invertible. $\square$

<u>Corollary:</u> The following are equivalent:
(1) λ is an eigenvalue of φ
(2) $\varphi - \lambda$ is not invertible
(3) $\lambda = \lambda_i$ for some diagonal entry of any upper-triangular matrix A representing φ.

((1)$\Leftrightarrow$(2) since eigenvectors $= \ker(\varphi - \lambda)$, and (2)$\Leftrightarrow$(3) by applying the lemma to $\varphi - \lambda$ and matrix $A - \lambda I$).

<u>Next goal</u>: further study of invariant subspaces & eigenvalues for linear operators over alg. closed k, especially $\mathbb{C}$ – <u>Jordan normal form</u>.

(This is Axler ch. 8 – we'll return to the skipped chapters 6 & 7 soon).

First we talk about <u>generalized eigenspaces</u>

Recall $\ker(\varphi) = \{v \in V / \varphi(v) = 0\}$.

<u>Def:</u> the <u>generalized kernel</u> of φ is $\mathrm{gker}(\varphi) = \{v \in V / \exists m > 0 \text{ st. } \varphi^m(v) = 0\}$

These are all the vectors that are eventually sent to 0 by repeatedly applying φ.

<u>Observe:</u> $0 \subset \ker\varphi \subset \ker(\varphi^2) \subset \dots$ (since: $\varphi^m(v) = 0 \Rightarrow \varphi^{m+1}(v) = 0 \dots$)
if $\ker(\varphi^m) = \ker(\varphi^{m+1})$ then the sequence remains constant after that!

(<u>Pf</u>: $\varphi^{m+2}(v) = \varphi^{m+1}(\varphi(v)) = 0 \Leftrightarrow \varphi(v) \in \ker\varphi^{m+1} = \ker\varphi^m \Leftrightarrow \varphi^m(\varphi(v)) = \varphi^{m+1}(v) = 0$. So $\ker(\varphi^{m+1}) = \ker(\varphi^{m+2})$)

Since the sequence stops increasing after at most $n = \dim V$ steps, $\mathrm{gker}(\varphi) = \ker\varphi^n$.

<u>Example:</u> $\varphi: k^2 \to k^2$, $e_1 \mapsto 0$, $e_2 \mapsto e_1$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ Then $\ker(\varphi) = k \cdot e_1$, but $\ker(\varphi^2) = \mathrm{gker}(\varphi) = k^2$.

Lemma: if $\mathrm{gker}(\varphi) = \ker(\varphi^m)$ then $V = \ker(\varphi^m) \oplus \mathrm{Im}(\varphi^m)$. using $\ker \varphi^m = \mathrm{gker}$.

Pf: If $v = \varphi^m(u) \in \mathrm{Im}(\varphi^m) \cap \ker(\varphi^m)$ then $\varphi^m(v) = \varphi^{2m}(u) = 0 \Rightarrow u \in \ker \varphi^{2m} \overset{\uparrow}{=} \ker \varphi^m$, so $v = \varphi^m(u) = 0$. Hence $\mathrm{Im}(\varphi^m) \cap \ker(\varphi^m) = \{0\}$. By dimension formula, $\mathrm{Im} \oplus \ker = V$. □

Def: Say φ is <u>nilpotent</u> if $\exists\, m > 0$ st. $\varphi^m = 0$, ie. $\mathrm{gker}(\varphi) = V$.

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* Now we can do the same thing to eigenspaces:

Def: $v \in V$ is a <u>generalized eigenvector</u> with generalized eigenvalue λ if $v \in \mathrm{gker}(\varphi - \lambda I)$ ie. $\exists\, m > 0$ st. $(\varphi - \lambda I)^m v = 0$. Call $\mathrm{gker}(\varphi - \lambda I)$ the <u>generalized eigenspace</u>

Def: The <u>multiplicity</u> of the eigenvalue λ is the dimension of the generalized eigenspace $V_\lambda = \mathrm{gker}(\varphi - \lambda I)$. $\left(= \ker(\varphi - \lambda I)^n\right)$.

In a basis where the matrix of φ is triangular, this is the number of times λ appears on the diagonal! (This will be clearer later...)

Prop.1: $V_\lambda = \ker(\varphi - \lambda I)^n$ and $W_\lambda = \mathrm{Im}(\varphi - \lambda I)^n$ are invariant subspaces of φ, and $V = V_\lambda \oplus W_\lambda$.

Proof:
- let $v \in V_\lambda$, then $(\varphi - \lambda I)^n v = 0$, hence $\varphi(\varphi - \lambda I)^n v = 0$. But $\varphi - \lambda I$ commutes with φ, so this implies $(\varphi - \lambda I)^n \varphi v = 0$, hence $\varphi(v) \in V_\lambda$.
- if $v = (\varphi - \lambda I)^n u \in W_\lambda$ then $\varphi(v) = \varphi(\varphi - \lambda I)^n u = (\varphi - \lambda I)^n \varphi(u) \in \mathrm{Im}(\varphi - \lambda I)^n = W_\lambda$.
- the lemma above, applied to $\varphi - \lambda I$, says $V = \ker(\varphi - \lambda I)^n \oplus \mathrm{Im}(\varphi - \lambda I)^n$. □

Prop 2: The subspaces $V_\lambda \subset V$ are independent: $\sum v_i = 0$, $v_i \in V_{\lambda_i} \Rightarrow v_i = 0\ \forall i$.

Proof: Assume $\sum_{i=1}^{\ell} v_i = 0$, $v_i \in V_{\lambda_i}$, λ_i distinct. We'll show $v_1 = 0$ (same for the others).
If $v_1 \neq 0$, let $k \geq 0$ be the largest integer st. $(\varphi - \lambda_1 I)^k v_1 = w \neq 0$ (but $(\varphi - \lambda_1 I)^{k+1} v = 0$, so $\varphi(w) = \lambda_1 w$).

Observe: $$(\varphi - \lambda_\ell I)^n \cdots (\varphi - \lambda_2 I)^n (\varphi - \lambda_1 I)^k \big(\overbrace{v_1 + \dots + v_\ell}^{=0}\big) = 0$$

is the sum of $(\varphi - \lambda_\ell I)^n \cdots (\varphi - \lambda_2 I)^n w = \prod_{j=2}^{\ell} (\lambda_1 - \lambda_j)^n w \neq 0$

and $(\varphi - \lambda_\ell I)^n \cdots (\varphi - \lambda_2 I)^n (\varphi - \lambda_1 I)^k v_j = 0 \quad \forall j \geq 2$

(because the operators $(\varphi - \lambda I)$ commute, and $(\varphi - \lambda_j I)^n v_j = 0$).

Contradiction, hence $v_1 = 0$, and similarly $v_i = 0\ \forall i$. □

Thm.: If k is alg. closed, V finite-dim. vect space over k, $\varphi: V \to V$, then V decomposes into the direct sum of the generalized eigenspaces V_λ of φ, $V = \bigoplus V_\lambda$.

Proof: By induction on $\dim V$! (the result is clear for $\dim V = 1$). Assume the result holds up to dimension $n-1$, and consider the case $\dim V = n$.

We've seen before: k alg. closed $\Rightarrow$ φ has at least one eigenvalue λ_1

Let $V_{\lambda_1} = \mathrm{gker}(\varphi - \lambda_1 I) = \ker((\varphi-\lambda_1 I)^n)$, $U = W_{\lambda_1} = \mathrm{Im}(\varphi - \lambda_1 I)^n$.

By prop. 1 above, V_{λ_1} and U are invariant subspaces, and $V = V_{\lambda_1} \oplus U$.

Since $\dim U < \dim V$, induction $\Rightarrow$ U decomposes into generalized eigenspaces for $\varphi_{|U}$,

$U = U_{\lambda_2} \oplus \dots \oplus U_{\lambda_\ell}$, $\lambda_2 \dots \lambda_\ell$ eigenvalues of $\varphi_{|U}$ ($\Leftrightarrow$ eigenvalues of φ with an eigenvector $\in U$)

$$U_{\lambda_j} = \ker(\varphi_{|U} - \lambda_j I)^n = \ker(\varphi - \lambda_j I)^n \cap U = V_{\lambda_j} \cap U$$

Moreover, $\varphi_{|U}$ doesn't have λ as eigenvalue (since $\ker(\varphi - \lambda I)^n \cap U = 0$), so $\lambda \notin \{\lambda_2 \dots \lambda_\ell\}$.

Now: $U_{\lambda_j} \subset \ker(\varphi - \lambda_j I)^n = V_{\lambda_j}$, and $V = V_{\lambda_1} \oplus U = V_{\lambda_1} \oplus U_{\lambda_2} \oplus \dots \oplus U_{\lambda_\ell}$.

Since the gen^d eigenspaces V_{λ_j} contain U_{λ_j} $\forall j \geq 2$, we find that $V_{\lambda_1}, \dots V_{\lambda_\ell}$ span V; and they are independent by Prop. 2, hence $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_\ell}$.

(and in fact $V_{\lambda_j} = U_{\lambda_j}$ $\forall j \geq 2$; in other terms, $\mathrm{Im}(\varphi - \lambda_i I)^n = \bigoplus_{j \neq i} \ker(\varphi - \lambda_j I)^n$.) $\square$

* The decomposition $V = \bigoplus V_{\lambda_i}$ gives us bases in which φ is given by a block diagonal matrix

$$\begin{pmatrix} \varphi_{|V_{\lambda_1}} & & 0 \\ & \varphi_{|V_{\lambda_2}} & \\ 0 & & \ddots \; \varphi_{|V_{\lambda_\ell}} \end{pmatrix}$$

* Moreover, $\varphi_{|V_{\lambda_i}}$ can be represented by a triangular matrix in a suitable basis for V_{λ_i} (using thm seen earlier), and since its only eigenvalue is λ_i, the diagonal entries are all λ_i! So: $\varphi \sim \begin{pmatrix} \begin{matrix}\lambda_1 & * \\ 0 & \lambda_1\end{matrix} & & 0 \\ & \begin{matrix}\lambda_2 & * \\ 0 & \lambda_2\end{matrix} & \\ 0 & & \ddots \; \begin{matrix}\lambda_\ell & * \\ 0 & \lambda_\ell\end{matrix} \end{pmatrix}$

* We can do more with the blocks $\begin{pmatrix} \lambda_i & * \\ 0 & \lambda_i \end{pmatrix}$ but this requires further study of nilpotent operators (note: $\varphi_{|V_{\lambda_i}} - \lambda_i I$ nilpotent!)

Nilpotent operators: let $\varphi: V \to V$ nilpotent (ie. $\varphi^m = 0$ for some $m \leq \dim V$).
(This part works over any field) **Goal**: find a "nice" basis of V for φ.

Observe: if $\dim V = 2$, then either $\varphi = 0$; or $\varphi^2 = 0$ but $\varphi \neq 0$.
In second case: let $v \notin \ker\varphi$, then $\varphi(v) = u \in \ker\varphi$ so u, v are independent and form a basis, in which $M(\varphi) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Jordan's method generalizes this to higher dimensions.