Last time: V finite dim. over k alg. closed (eg. $\mathbb{C}$), $\varphi: V \to V$ linear operator $\Rightarrow$

- $\exists$ basis st. $\mathcal{M}(\varphi)$ is upper triangular $\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$
- $\varphi - \lambda I$ is invertible $\Leftrightarrow \lambda \notin \{\lambda_1 \dots \lambda_n\}$, so the diagonal entries are the eigenvalues of φ!
- the eigenspaces $\mathrm{Ker}(\varphi - \lambda_i)$ are linearly independent, but need not span V (if they do: $\exists$ basis of eigenvectors, hence φ is diagonalizable)
- To do better, we introduced the <u>generalized eigenspaces</u>

$$V_\lambda = \{v \in V \mid \exists m \in \mathbb{N} \text{ st. } (\varphi - \lambda)^m v = 0\} = g\mathrm{ker}(\varphi - \lambda) \underset{\uparrow}{=} \mathrm{Ker}(\varphi - \lambda)^n$$

(This is only nontrivial if λ is an eigenvalue of φ) $\quad$ $\mathrm{ker}(\varphi-\lambda) \subset \mathrm{ker}(\varphi-\lambda)^2 \subset \dots$ becomes constant in at most $n = \dim V$ steps

Prop. 1: $V_\lambda = \mathrm{ker}(\varphi - \lambda I)^n$ and $W_\lambda = \mathrm{Im}(\varphi - \lambda I)^n$ are invariant subspaces of φ, and $V = V_\lambda \oplus W_\lambda$.

Prop 2: The subspaces $V_\lambda \subset V$ are independent: $\sum v_i = 0,\ v_i \in V_{\lambda_i} \Rightarrow v_i = 0\ \forall i$.

Thm.: If k is alg. closed, V finite-dim. vect space over k, $\varphi: V \to V$, then V decomposes into the direct sum of the generalized eigenspaces V_λ of φ, $V = \bigoplus V_\lambda$.

Proof: By induction on $\dim V$! (the result is clear for $\dim V = 1$). Assume the result holds up to dimension $n-1$, and consider the case $\dim V = n$.

We've seen before: k alg. closed $\Rightarrow \varphi$ has at least one eigenvalue λ_1

Let $V_{\lambda_1} = g\mathrm{ker}(\varphi - \lambda_1 I) = \mathrm{ker}((\varphi - \lambda_1 I)^n)$, $U = W_{\lambda_1} = \mathrm{Im}(\varphi - \lambda_1 I)^n$.

By prop. 1 above, V_{λ_1} and U are invariant subspaces, and $V = V_{\lambda_1} \oplus U$.

Since $\dim U < \dim V$, induction $\Rightarrow$ U decomposes into generalized eigenspaces for $\varphi_{|U}$,

$U = U_{\lambda_2} \oplus \dots \oplus U_{\lambda_\ell}$, $\lambda_2 \dots \lambda_\ell$ eigenvalues of $\varphi_{|U}$ ($\Leftrightarrow$ eigenvalues of φ with an eigenvector $\in U$)

$$U_{\lambda_j} = \mathrm{ker}(\varphi_{|U} - \lambda_j I)^n = \mathrm{ker}(\varphi - \lambda_j I)^n \cap U = V_{\lambda_j} \cap U$$

Moreover, $\varphi_{|U}$ doesn't have λ as eigenvalue (since $\mathrm{Ker}(\varphi - \lambda I)^n \cap U = 0$), so $\lambda \notin \{\lambda_2 \dots \lambda_\ell\}$.

Now: $U_{\lambda_j} \subset \mathrm{Ker}(\varphi - \lambda_j I)^n = V_{\lambda_j}$, and $V = V_{\lambda_1} \oplus U = V_{\lambda_1} \oplus U_{\lambda_2} \oplus \dots \oplus U_{\lambda_\ell}$.

Since the gen^d eigenspaces V_{λ_j} contain U_{λ_j} $\forall j \geq 2$, we find that $V_{\lambda_1} \dots V_{\lambda_\ell}$ span V; and they are independent by Prop. 2, hence $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_\ell}$.

(and in fact $V_{\lambda_j} = U_{\lambda_j}$ $\forall j \geq 2$; in other terms, $\mathrm{Im}(\varphi - \lambda_i I)^n = \bigoplus_{j \neq i} \mathrm{ker}(\varphi - \lambda_j I)^n$.) $\square$

②

* The decomposition $V = \bigoplus V_{\lambda_i}$ gives us bases in which φ is given by a block diagonal matrix $\begin{pmatrix} \varphi_{|V_{\lambda_1}} & & & 0 \\ & \varphi_{|V_{\lambda_2}} & & \\ & & \ddots & \\ 0 & & & \varphi_{|V_{\lambda_\ell}} \end{pmatrix}$

* Moreover, $\varphi_{|V_{\lambda_i}}$ can be represented by a triangular matrix in a suitable basis for V_{λ_i} (using thm seen last time), and since its only eigenvalue is λ_i, the diagonal entries are all λ_i! So: $\varphi \sim \begin{pmatrix} \begin{matrix}\lambda_1 & * \\ 0 & \lambda_1\end{matrix} & & & 0 \\ & \boxed{\begin{matrix}\lambda_2 & * \\ 0 & \lambda_2\end{matrix}} & & \\ & & \ddots & \\ 0 & & & \begin{matrix}\lambda_\ell & * \\ 0 & \lambda_\ell\end{matrix} \end{pmatrix}$

* We can do more with the blocks $\begin{pmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix}$ but this requires further study of <u>nilpotent operators</u> (note: $\varphi_{|V_{\lambda_i}} - \lambda_i I$ nilpotent!)

<u>Def</u>: $\varphi : V \to V$ is <u>nilpotent</u> if $\exists\, m \in \mathbb{N}$ st. $\varphi^m = 0$ ie. $gKer(\varphi) = V$.
($\iff \varphi^n = 0$ for $n = \dim V$).

<u>Goal</u>: find a "nice" basis of V for a nilpotent operator $\varphi : V \to V$.
(This works over <u>any</u> field, don't need to alg. closed).

<u>Observe</u>: if $\dim V = 2$, there are 2 cases: either $\varphi = 0$; or $\varphi^2 = 0$ but $\varphi \neq 0$.
In second case: let $v \notin \ker\varphi$, then $\varphi(v) = u \in \ker\varphi$ so u, v are independent and form a basis, in which $\mathcal{M}(\varphi) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Jordan's method generalizes this to higher dimensions:

<u>Prop</u>: $\exists$ basis of V: $\{\varphi^{m_1}(v_1), \varphi^{m_1-1}(v_1), \dots, v_1, \quad \dots, \quad \varphi^{m_k}(v_k), \dots, v_k\}$ where $\varphi^{m_i+1}(v_i) = 0 \quad \forall i$
in which $\mathcal{M}(\varphi) = \begin{pmatrix} \begin{matrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{matrix} & & & 0 \\ & \boxed{\begin{matrix} 0 & 1 & \\ & \ddots & 1 \\ & & 0 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \begin{matrix} 0 & 1 & \\ & \ddots & 1 \\ & & 0 \end{matrix} \end{pmatrix}$ block diagonal built from <u>nilpotent Jordan blocks</u> $\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}$
(each basis element $\mapsto$ previous one)
first basis elt $\mapsto 0$

<u>Proof</u>: Recall $0 \subset \ker\varphi \subset \ker\varphi^2 \subset \dots \subset \ker\varphi^m = V$. assume this is the smallest m, ie. $\varphi^m = 0$ but $\varphi^{m-1} \neq 0$.

* Claim: if a subspace $U \subset \ker(\varphi^{k+1})$ satisfies $\ker(\varphi^k) \cap U = \{0\}$ ($k \geq 1$), then $\varphi_{|U}$ is injective, $\varphi(U) \subset \ker(\varphi^k)$, and $\ker(\varphi^{k-1}) \cap \varphi(U) = \{0\}$.

Indeed: $\forall\, v \in U,\ v \neq 0 \Rightarrow \begin{cases} \varphi^k(v) \neq 0. \\ \varphi^{k+1}(v) = 0 \end{cases}$ In particular $\varphi(v) \neq 0$, ie. $\ker(\varphi_{|U}) = \{0\}$, injective.
Also, $\varphi^k(\varphi(v)) = 0 \Rightarrow \varphi(v) \in \ker\varphi^k$
and $\varphi^{k-1}(\varphi(v)) = \varphi^k(v) \neq 0 \Rightarrow \varphi(v) \notin \ker\varphi^{k-1}$!

* First step: let U_m st. $\ker(\varphi^m) = V = \ker(\varphi^{m-1}) \oplus U_m$
& pick a basis $(v_{m,1}, \dots, v_{m,k_m})$ of U_m [these will yield Jordan blocks of size m!]

(eg: start from a basis of $\ker \varphi^{m-1}$, extend to basis of V by adding vectors $v_{m,1},\dots,v_{m,k_m}$, and let U_m be their span.)

Now by the claim, $v_{m-1,1}=\varphi(v_{m,1}),\dots, v_{m-1,k_m}=\varphi(v_{m,k_m})$ are linearly independent, and their span is $\subset \ker(\varphi^{m-1})$ but independent of $\ker(\varphi^{m-2})$.
Start from a basis of $\ker(\varphi^{m-2})$, add $v_{m-1,1},\dots,v_{m-1,k_m}$ and complete to a basis of $\ker(\varphi^{m-1})$ by adding some other vectors $v_{m-1,k_m+1},\dots,v_{m-1,k_{m-1}}$ (if needed: could have $k_{m-1}=k_m$). (these will yield blocks of size $m-1$).
Let $U_{m-1}=\mathrm{span}(v_{m-1,1},\dots,v_{m-1,k_{m-1}})$. Then $\ker(\varphi^{m-1})=\ker(\varphi^{m-2})\oplus U_{m-1}$.
And so on: given $U_j=\mathrm{span}(v_{j,1}\dots v_{j,k_j})$ with $\ker\varphi^j=\ker\varphi^{j-1}\oplus U_j$, take $v_{j-1,i}=\varphi(v_{j,i})$ for $1\le i\le k_j$ and extend by adding vectors as needed to build U_{j-1}. This eventually gives a basis of $V=U_1\oplus\dots\oplus U_m$, and rearranging it as $(v_{1,1},\dots,v_{m,1},v_{1,2},\dots)$ we get the result. $\square$

We now combine our results to arrive at the

<u>Jordan normal form</u>: V finite dim. vector space over k alg. closed (eg. $\mathbb{C}$), $\varphi\in \mathrm{Hom}(V,V)$ $\Rightarrow \exists$ basis of V in which the matrix of φ is block-diagonal, with each block a <u>Jordan block</u> $\begin{pmatrix}\lambda & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda\end{pmatrix}$.

Rmks: • size 1 Jordan block: (λ), size 2: $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$, ... φ is diagonalizable $\Leftrightarrow$ all the blocks have size 1.
- the values of λ that appear are exactly the eigenvalues of φ. There may be several blocks with the same λ; their direct sum is the generalized eigenspace V_λ.
- proof: we've seen $V=\bigoplus V_\lambda$ generalized eigenspaces; now $\varphi_{|V_\lambda}-\lambda I$ is nilpotent, so can decompose into nilpotent Jordan blocks $\varphi_{|V_\lambda}-\lambda I=\bigoplus\begin{pmatrix}0 & 1 & \\ & \ddots & 1\\ & & 0\end{pmatrix}$, so $\varphi_{|V_\lambda}=\bigoplus\begin{pmatrix}\lambda & 1 & \\ & \ddots & 1\\ & & \lambda\end{pmatrix}$.

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* <u>Characteristic polynomial, minimal polynomial</u>:

Let k be algebraically closed, $\varphi: V\to V$, $V=\bigoplus_{i=1}^{\ell} V_{\lambda_i}$ V_{λ_i} generalized eigenspaces
Call • $n_i=\dim V_{\lambda_i}$ the <u>multiplicity</u> of λ_i $(\sum n_i=\dim V)$
• $m_i=$ nilpotence order of $(\varphi_{|V_{\lambda_i}}-\lambda_i Id)$ ie. smallest m_i st. $V_{\lambda_i}=\ker(\varphi-\lambda_i I)^{m_i}$

From the above: $m_i\le n_i$, and V_{λ_i} is diagonalizable iff all $m_i=1$.

<u>Def</u>: • The <u>characteristic polynomial</u> of φ is $\chi_\varphi(x)=\prod_{i=1}^{\ell}(x-\lambda_i)^{n_i}$

The usual definition, once we have defined determinant, is: $\chi_\varphi(x)=\det(xI-\varphi)$.

Manifestly, in a basis where $\mathcal{M}(\varphi)$ is triangular (or Jordan), $\mathcal{M}(xI-\varphi)=\begin{pmatrix} x-\lambda_1 & & * \\ & \ddots & \\ & & x-\lambda_n \end{pmatrix}$ and this is the same thing. (but can use <u>any</u> basis to calculate det).

The significance is: given matrix of φ in any basis, A, we can calculate $\chi(x) = \det(xI-A) \in k[x]$, and solve for roots = eigenvalues
multiplicities = dim. of gen^d eigenspaces.
(This also works over non alg. closed k, without any guarantee that $\chi(x)$ has any roots.)

<u>Def</u>: The <u>minimal polynomial</u> of φ is $\mu_\varphi(x) = \prod_{i=1}^{\ell} (x-\lambda_i)^{m_i}$.

<u>Significance</u>: $(\varphi-\lambda_i)^k = 0$ on the gen. eigenspace V_{λ_i} iff $k \geq m_i$
& invertible on the other gen^d eigenspaces.
So $\mu_\varphi(\varphi)$ = simplest polynomial expression in φ that is zero on all V_{λ_i}'s, hence on $\bigoplus V_{\lambda_i} = V$.

Hence: $\mu_\varphi(\varphi) = 0$, and $\forall p \in k[x]$, $p(\varphi) = 0 \in \mathrm{Hom}(V,V)$ iff μ_φ divides p.

Since nilpotence order m_i is always $\leq \dim V_{\lambda_i} = n_i$, μ_φ divides χ_φ, so:

<u>Thm</u> (Cayley-Hamilton) $\chi_\varphi(\varphi) = 0$.

(This is also true over non alg. closed k, by passing to alg. closure; see below for an example)

- A word about operators on finite dim. <u>$\mathbb{R}$-vector spaces</u>:

Let V real vector space (dim. n), $\varphi: V \to V$ linear operator.
Since $\mathbb{R}$ is not alg. closed, φ might not have eigenvalues, and we can't put φ in triangular or Jordan form.

<u>Yet</u>: every real operator has an invariant subspace of dim. $\underline{1}$ or 2

<u>Approach</u>: work over $\mathbb{C}$ which is alg. closed. How do we do this?

<u>Def</u>: The <u>complexification</u> of V is $V_\mathbb{C} = V \times V = \{v + iw \mid v, w \in V\}$,
with addition $(v_1 + iw_1) + (v_2 + iw_2) = (v_1+v_2) + i(w_1+w_2)$
scalar mult. $(a+ib)(v+iw) = (av - bw) + i(bv + aw)$
$a, b \in \mathbb{R}$

- This is a $\mathbb{C}$-vector space of dimension n: if $(e_1, \dots e_n)$ is a basis of V over $\mathbb{R}$, then $e_1 (= e_1 + i0), \dots, e_n$ is also a basis of $V_\mathbb{C}$ over $\mathbb{C}$.

- Given $\varphi : V \to V$ $\mathbb{R}$-linear, we can extend it to $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ $\mathbb{C}$-linear simply by $\varphi_{\mathbb{C}}(v+iw) = \varphi(v) + i\varphi(w)$. Choosing a basis $(e_1 \dots e_n)$ as above, the matrix of $\varphi_{\mathbb{C}}$ is the <u>same</u> as that of φ $\left(\varphi_{\mathbb{C}}(e_j + i0) = \varphi(e_j) + i0\right)$.

But now... $\varphi_{\mathbb{C}}$ is guaranteed to have an eigenvector!

(and gen.^d eigenspaces, and Jordan form, ...)

Let $\mathbb{v} = v + iw$ be an eigenvector of $\varphi_{\mathbb{C}}$ for eigenvalue $\lambda \in \mathbb{C}$, $\varphi_{\mathbb{C}}(\mathbb{v}) = \lambda \mathbb{v}$.

There are two cases:

- if $\lambda \in \mathbb{R}$, then $\varphi_{\mathbb{C}}(v+iw) = \varphi(v) + i\varphi(w) = \lambda v + i\lambda w$

  $\Rightarrow$ $v = \mathrm{Re}(\mathbb{v})$ and $w = \mathrm{Im}(\mathbb{v})$ are eigenvectors of φ with the same eigenvalue λ (if they are nonzero; one of them is).

  (⚠ the multiplicity of λ for φ has no reason to be even).

- if $\lambda = a + ib \notin \mathbb{R}$, then $\varphi_{\mathbb{C}}(v+iw) = (a+ib)(v+iw)$

  $\Rightarrow \varphi_{\mathbb{C}}(v - iw) = (a-ib)(v-iw)$ (compare real and imaginary parts!)

  i.e. $\overline{\mathbb{v}} = v - iw$ is an eigenvector of $\varphi_{\mathbb{C}}$ with eigenvalue $\overline{\lambda}$.

  It follows that v and w are linearly independent, and span a 2-dimensional invariant subspace $U \subset V$:

$$\begin{aligned} \varphi(v) &= av - bw \\ \varphi(w) &= bv + aw \end{aligned} \qquad \mathcal{M}(\varphi_{|U}, (v,w)) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

(One could further study block triangular decompositions of φ etc. starting from $\varphi_{\mathbb{C}}$).