Math 55a Homework 5

Due Wednesday October 6, 2021.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: Generalized eigenvectors, nilpotent operators, Jordan normal form; categories and functors; bilinear forms, inner product spaces, orthogonality. (Axler chapters 6 and 8; Artin §4.4-4.7 and 8.1-8.5; handout on categories and functors.)

0. Sometime over the weekend of October 2-3, please complete the week 5 survey (in Canvas).

1. Let V be an n-dimensional vector space, and $T : V \to V$ a linear map; suppose that T is nilpotent, i.e. $T^N = 0$ for some $N > 0$.

(a) Show that $T^n = 0$.

(b) Show that $I + T$ is invertible (where I is the identity operator), and give a formula for its inverse.

2. Let V be a vector space over a field k and $\phi : V \to V$ a linear operator.

(a) Show that for all $m \geq 1$, $\mathrm{Im}(\phi^{m+1}) \subset \mathrm{Im}(\phi^m)$, and if $\mathrm{Im}(\phi^{m+1}) = \mathrm{Im}(\phi^m)$ then $\mathrm{Im}(\phi^n) = \mathrm{Im}(\phi^m)$ for all $n \geq m$.

(b) The *eventual image* of ϕ, denoted $\mathrm{evIm}(\phi)$, is the set of vectors which can be expressed as $\phi^m(v)$ for *all* $m \in \mathbb{N}$, i.e. $\mathrm{evIm}(\phi) = \bigcap_{m\geq 1} \mathrm{Im}(\phi^m)$. Show that $\mathrm{evIm}(\phi)$ is an invariant subspace for ϕ, and that if V is finite-dimensional then the restriction of ϕ to $\mathrm{evIm}(\phi)$ is surjective.

(c) Show that, if V is finite-dimensional, then the eventual image of ϕ and its generalized kernel $\mathrm{gKer}(\phi) = \{v \in V \mid \exists m \in \mathbb{N},\ \phi^m(v) = 0\}$ coincide with the image and kernel of ϕ^n where $n = \dim V$, and give a direct sum decomposition $V = \mathrm{evIm}(\phi) \oplus \mathrm{gKer}(\phi)$, where ϕ is invertible on $\mathrm{evIm}(\phi)$ and nilpotent on $\mathrm{gKer}(\phi)$.

(d) Show that, if V is infinite-dimensional, then none of the statements in (c) need to hold: find an infinite-dimensional vector space V and two linear operators $\phi, \psi : V \to V$ for which:

1. $\mathrm{evIm}(\phi) = \mathrm{gKer}(\phi) = V$, the restriction of ϕ to $\mathrm{evIm}(\phi)$ is not injective, and the restriction of ϕ to $\mathrm{gKer}(\phi)$ is not nilpotent;
2. $\mathrm{evIm}(\psi) = \mathrm{gKer}(\psi) = 0$.

3. Fix a field k, and consider the category Vect_k of all vector spaces over k.

(a) Show that there exists a contravariant functor from the category Vect_k to itself, which on objects takes each vector space V to its dual $V^* = \mathrm{Hom}(V, k)$.

(b) Recall that for each vector space V we have a "natural" homomorphism $ev_V : V \to V^{**}$ taking every vector $v \in V$ to the element $ev_V(v)$ of $V^{**} = \mathrm{Hom}(V^*, k)$ which maps $\ell \in V^*$ to $\ell(v) \in k$.

Show that these homomorphisms determine a natural transformation from the identity functor to the square of the functor of part (a).

4. Suppose the rows of an $n \times n$ real matrix $A \in M_n(\mathbb{R})$ form an orthonormal basis for $\mathbb{R}^n$ with its usual inner product. Show that the same is true of the columns of A.

(Hint: consider the product of A with its transpose.)

5. Let $V \subset \mathbb{R}[x]$ be the space of real polynomials of degree at most 2, and define an inner product on V by $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$. Find an orthonormal basis of V with respect to this inner product.

6. Let V be a finite-dimensional vector space over a field k, and $b : V \times V \to k$ a non-degenerate bilinear form. A subspace $\Lambda \subset V$ is called *isotropic* for b if $b(v, w) = 0\ \forall v, w \in \Lambda$. Show that, if Λ is isotropic, then $\dim(\Lambda) \leq \frac{1}{2}\dim(V)$.

7. Let V be an n-dimensional vector space over a field k, with $\mathrm{char}(k) \neq 2$. The set of bilinear forms $V \times V \to k$ has the structure of a vector space $B(V)$, of dimension n^2. We say that a bilinear form is *symmetric* if $b(v, w) = b(w, v)$ for all $v, w \in V$, *skew-symmetric* if $b(v, w) = -b(w, v)$ for all $v, w \in V$.

(a) Show that the subset $B_{symm}(V) \subset B(V)$ of symmetric bilinear forms is a subspace of $B(V)$, and calculate its dimension.

(b) Similarly, show that the subset $B_{skew}(V) \subset B(V)$ of skew-symmetric bilinear forms is a subspace of $B(V)$ and calculate its dimension.

(c) Show that $B(V) = B_{symm}(V) \oplus B_{skew}(V)$. Does this remain true if $\mathrm{char}(k) = 2$?

8.* Let V be an n-dimensional vector space over a field k, with $\mathrm{char}(k) \neq 2$.

(a) How should one define the notion of a *trilinear form* $t : V \times V \times V \to k$? Show that the set $T(V)$ of trilinear forms on V can be given the structure of a vector space, and calculate its dimension.

(b) Say a trilinear form $t : V \times V \times V \to k$ is *symmetric* if the value $t(u, v, w)$ is unchanged if we permute the variables, and denote by $T_{symm}(V) \subset T(V)$ the subset of symmetric trilinear forms. Show that $T_{symm}(V)$ is a subspace of $T(V)$, and calculate its dimension.

(c) Say a trilinear form $t : V \times V \times V \to k$ is *skew-symmetric* (or *alternating*) if permuting the variables multiplies the value $t(u, v, w)$ by ± 1 according to the sign of the permutation. Show that the subset $T_{skew}(V) \subset T(V)$ of skew-symmetric trilinear forms is a subspace of $T(V)$, and calculate its dimension.

(d) Show that, if $n \geq 2$, then we do *not* have $T(V) = T_{symm}(V) \oplus T_{skew}(V)$. (The reason for this will become clearer when we discuss the representation theory of the symmetric group.)

9.* (Optional, extra credit) Let $S_1, \dots, S_m$ be subsets of $\{1, \dots, n\}$ such that S_i contains an odd number of elements for each $1 \leq i \leq m$, and $S_i \cap S_j$ contains an even number of elements for each $1 \leq i < j \leq m$. Show that $m \leq n$.

(Hint: construct vectors $v_1, \dots, v_m$ in $(\mathbb{F}_2)^n$, and use the standard dot product (mod 2) to show they are linearly independent).

10. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?