- A word about operators on finite dim. $\mathbb{R}$-vector spaces:

Let V real vector space (dim. n), $\varphi: V \to V$ linear operator.
Since $\mathbb{R}$ is not alg. closed, φ might not have eigenvalues, and we can't put φ in triangular or Jordan form.

Yet: every real operator has an invariant subspace of dim. 1 or 2

Approach: work over $\mathbb{C}$ which is alg. closed. How do we do this?

Def: The complexification of V is $V_{\mathbb{C}} = V \times V = \{v + iw \mid v, w \in V\}$,
with addition $(v_1 + iw_1) + (v_2 + iw_2) = (v_1 + v_2) + i(w_1 + w_2)$
scalar mult. $(a+ib)(v+iw) = (av - bw) + i(bv + aw)$
$a, b \in \mathbb{R}$

- This is a $\mathbb{C}$-vector space of dimension n: if $(e_1 \dots e_n)$ is a basis of V over $\mathbb{R}$, then $e_1 (= e_1 + i0), \dots, e_n$ is also a basis of $V_{\mathbb{C}}$ over $\mathbb{C}$.

- Given $\varphi: V \to V$ $\mathbb{R}$-linear, we can extend it to $\varphi_{\mathbb{C}}: V_{\mathbb{C}} \to V_{\mathbb{C}}$ $\mathbb{C}$-linear simply by $\varphi_{\mathbb{C}}(v + iw) = \varphi(v) + i\varphi(w)$. Choosing a basis $(e_1 \dots e_n)$ as above, the matrix of $\varphi_{\mathbb{C}}$ is the same as that of φ $\left(\varphi_{\mathbb{C}}(e_j + i0) = \varphi(e_j) + i0\right)$.

But now... $\varphi_{\mathbb{C}}$ is guaranteed to have an eigenvector!
(and gen.d eigenspaces, and Jordan form, ...)

Let $\mathbb{v} = v + iw$ be an eigenvector of $\varphi_{\mathbb{C}}$ for eigenvalue $\lambda \in \mathbb{C}$, $\varphi_{\mathbb{C}}(\mathbb{v}) = \lambda \mathbb{v}$.
There are two cases:

- if $\lambda \in \mathbb{R}$, then $\varphi_{\mathbb{C}}(v + iw) = \varphi(v) + i\varphi(w) = \lambda v + i\lambda w$
  $\Rightarrow v = \mathrm{Re}(\mathbb{v})$ and $w = \mathrm{Im}(\mathbb{v})$ are eigenvectors of φ with the same eigenvalue λ (if they are nonzero; one of them is).
  (⚠ the multiplicity of λ for φ has no reason to be even).
- if $\lambda = a + ib \notin \mathbb{R}$, then $\varphi_{\mathbb{C}}(v + iw) = (a+ib)(v+iw)$
  $\Rightarrow \varphi_{\mathbb{C}}(v - iw) = (a - ib)(v - iw)$ (compare real and imaginary parts!)
  ie. $\bar{\mathbb{v}} = v - iw$ is an eigenvector of $\varphi_{\mathbb{C}}$ with eigenvalue $\bar{\lambda}$.
  It follows that v and w are linearly independent, and span a 2-dimensional invariant subspace $U \subset V$: $\varphi(v) = av - bw$, $\varphi(w) = bv + aw$ $\qquad \mathcal{M}(\varphi_{|U}, (v, w)) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

(One could further study block triangular decompositions of φ etc. starting from $\varphi_{\mathbb{C}}$).

Interlude: the language of categories. (then we'll return to (bi)linear algebra)

Def: A category is a collection of objects + for each pair of objects, a collection of morphisms $\mathrm{Mor}(A,B)$, and an operation called composition of morphisms,
$$\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \to \mathrm{Mor}(A,C) \quad \text{st.}$$
$$f, \; g \longmapsto g \circ f$$
1) every object A has an identity morphism $id_A \in \mathrm{Mor}(A,A)$ st. $\forall f \in \mathrm{Mor}(A,B)$, $f \circ id_A = id_B \circ f = f$.

2) composition is associative: $(f\circ g)\circ h = f \circ (g \circ h)$.

Ex:
1) category of sets, $\mathrm{Mor}(A,B)$ = all maps $A \to B$
2) Vect_k finite-dim'l vector spaces$/K$, linear maps.
3) groups, group homomorphisms
4) top. spaces, continuous maps.

Def: $f \in \mathrm{Mor}(A,B)$ is an isomorphism if $\exists g \in \mathrm{Mor}(B,A)$ st. $g\circ f = id_A$ and $f \circ g = id_B$.
(the inverse isomorphism)

Check:
- the inverse of f, if it exists, is unique..
- id_A is an isomorphism; f iso $\Rightarrow f^{-1}$ iso; f, g isos $\Rightarrow g\circ f$ isos.

$\leadsto$ The automorphisms of A, $\mathrm{Aut}(A) = \{\text{isomorphisms } A \to A\} \subset \mathrm{Mor}(A,A)$, form a group.

- Isomorphic objects have isomorphic automorphism groups: an isomorphism $f \in \mathrm{Mor}(A,B)$ determines an isom. of groups $c_f : \mathrm{Aut}(A) \to \mathrm{Aut}(B)$, $g \mapsto f\circ g \circ f^{-1}$.

Ex:
1) In Sets, A finite set with n elements $\Rightarrow \mathrm{Aut}(A) = \{\text{bijections } A \to A\} \cong \mathfrak{S}_n$
2) V = n-dim'l vector space$/k$ $\Rightarrow \mathrm{Aut}(V) \cong GL_n(k)$ invertible $n\times n$ matrices

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* Products and sums in categories:

- Given objects A, B in a category $\mathcal{C}$, a product $A \times B$ is an object Z of $\mathcal{C}$ and a pair of maps $\pi_1 : Z \to A$, $\pi_2 : Z \to B$ st. $\forall T \in \mathrm{ob}\,\mathcal{C}$, $\forall f_1 \in \mathrm{Mor}(T,A)$, $f_2 \in \mathrm{Mor}(T,B)$, $\exists !$ (unique) $\varphi \in \mathrm{Mor}(T,Z)$ st. $\pi_1 \circ \varphi = f_1$ and $\pi_2 \circ \varphi = f_2$.

$$\begin{tikzcd} & T \arrow[dl, bend right, "f_1"'] \arrow[d, dashed, "\exists! \varphi"] \arrow[dr, bend left, "f_2"] & \\ A & Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Ex. in Sets, $Z = A \times B$ usual Cartesian product
π_1, π_2 projection maps
given $f_1 : T \to A$, $f_2 : T \to B$, def. φ: $\begin{array}{l} T \to A \times B \\ t \mapsto (f_1(t), f_2(t)) \end{array}$

Same in Groups, Vect_k

- A *sum* of objects A and B is an object Z of $\mathcal{C}$ + maps $i_1 : A \to Z$, $i_2 : B \to Z$ s.t. $\forall T \in \mathrm{ob}\,\mathcal{C}$, $\forall f_1 \in \mathrm{Mor}(A,T)$, $\forall f_2 \in \mathrm{Mor}(B,T)$,
  $\exists!\ \varphi \in \mathrm{Mor}(Z,T)$ s.t. $\varphi \circ i_1 = f_1$ & $\varphi \circ i_2 = f_2$.

$$\begin{tikzcd} & T & \\ A \arrow[ur, bend left, "f_1"] \arrow[r, "i_1"'] & Z \arrow[u, dashed, "\varphi"] & B \arrow[l, "i_2"] \arrow[ul, bend right, "f_2"'] \end{tikzcd}$$

<u>Ex:</u> in Sets, this is $Z = A \sqcup B$ disjoint union; define $\varphi : Z \to T$
$$x \mapsto \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in B. \end{cases}$$

in Vect_k, it's $Z = A \oplus B$ (so... sum = product!)
with i_1, i_2 = inclusion of A as $A \oplus 0 \subset Z$, B as $0 \oplus B \subset Z$; define $\varphi : Z \to T$, $(a,b) \mapsto f_1(a) + f_2(b)$.

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* <u>Functors</u>:

<u>Def</u>: C, D categories. A (covariant) *functor* $F : C \to D$ is an assignment
- to each object X in C, an object $F(X)$ in D.
- to each morphism $f \in \mathrm{Mor}_C(X,Y)$, a morphism $F(f) \in \mathrm{Mor}_D(F(X), F(Y))$

s.t. 1) $F(id_X) = id_{F(X)}$
2) $F(g \circ f) = F(g) \circ F(f)$.

<u>Ex</u>: 1) *forgetful functor* taking a group, a top. space, ... to the underlying set.
2) on vector spaces, given a vect. space V, $F : W \longmapsto \mathrm{Hom}(V, W)$
if $f : W \to W'$ is linear, then induced map $\mathrm{Hom}(V,W) \xrightarrow{F(f)} \mathrm{Hom}(V,W')$, $a \longmapsto f \circ a$.
This gives a functor $\mathrm{Vect}_k \to \mathrm{Vect}_k$ (denoted $\mathrm{Hom}(V, \cdot)$)
3) Complexification, $\mathrm{Vect}_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{C}}$: on objects, $V \mapsto V_{\mathbb{C}}$, on morphisms $\varphi \mapsto \varphi_{\mathbb{C}}$ seen last time

* A *contravariant functor* = same except direction of morphisms is reversed:
$f \in \mathrm{Mor}_C(X,Y) \mapsto F(f) \in \mathrm{Mor}_D(F(Y), F(X))$; $F(g \circ f) = F(f) \circ F(g)$.

<u>Ex</u>: on Vect_k, $V \mapsto V^*$ dual (see HW5).

* There's one more layer to this, if you love category theory: given 2 functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* t from F to G is the data, $\forall X \in \mathrm{ob}\,\mathcal{C}$, of a morphism $t_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X))$, s.t. $\forall X, Y \in \mathrm{ob}\,\mathcal{C}$, $\forall f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$,

$$\begin{tikzcd} F(X) \arrow[r, "t_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "t_Y"'] & G(Y) \end{tikzcd}$$

commutes in $\mathcal{D}$.

Ex: on Vect_k, $V \mapsto V^{**}$ double dual is a (covariant) functor. We've said there is a "natural" map $ev_V : V \to V^{**}$ (isom. if $\dim < \infty$)
$v \mapsto (\ell \mapsto \ell(v))$

The precise meaning is: ev_V is part of a natural transformation of functors $\mathrm{Vect}_k \to \mathrm{Vect}_k$, from the identity functor to the double dual functor.
(see HW 5)

Bilinear forms:

Def: A bilinear form on a vector space V over field k is a map $b: V \times V \to k$ that is linear in each variable: $\forall u, v, w \in V$, $\forall \lambda \in k$,
$$\begin{cases} b(\lambda v, w) = b(v, \lambda w) = \lambda b(v,w) \\ b(u+v, w) = b(u,w) + b(v,w) \\ b(u, v+w) = b(u,v) + b(u,w). \end{cases}$$

This is not a linear map $V \times V \to k$ $\quad (b(\lambda(v,w)) = b(\lambda v, \lambda w) = \lambda^2 b(v,w) \neq \lambda b(v,w))$.

Def: We say b is symmetric if $b(v,w) = b(w,v)$ $\forall v, w \in V$
skew-symmetric if $b(v,w) = -b(w,v)$

Ex:
- the usual dot product on k^n, $(v,w) \mapsto \sum_{i=1}^n v_i w_i$ is symmetric.
- $b: k^2 \times k^2 \to k$, $b((x_1,x_2),(y_1,y_2)) = x_1 y_2 - x_2 y_1$ $\left(= \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right)$ is skew symmetric

* Given a bilinear map $b: V \times V \to k$, we get a linear map $\varphi_b : V \to V^*$ by defining $\varphi_b(v) = b(v, \cdot) \in V^*$ (maps $w \in V$ to $b(v,w) \in k$).
Conversely, $\varphi: V \to V^*$ determines $b(v,w) = (\varphi(v))(w)$ bilinear form.
This defines a bijection $B(V) \xrightarrow{\sim} \mathrm{Hom}(V, V^*)$.

Def: The rank of $b: V \times V \to k$ is the rank of $\varphi_b : V \to V^*$ $(= \dim \mathrm{Im}\, \varphi_b)$.
If φ_b is an isomorphism, say b is nondegenerate.

* For a given vector space V, $B(V) = \{\text{bilinear forms } V \times V \to k\}$ is a vector space over k. What is its dimension?
If we choose a basis $\{e_1 \dots e_n\}$ for V, it is enough to specify $b(e_i, e_j)$ $\forall i,j$ in order to determine b: by bilinearity, $b\left(\sum_i x_i e_i, \sum_j y_j e_j\right) = \sum_{i,j} x_i y_j\, b(e_i, e_j)$.
The values of $b(e_i, e_j)$ can be chosen freely - eg. a basis of $B(V)$ is given by $(b_{k\ell})_{\substack{1 \le k \le n \\ 1 \le \ell \le n}}$ $\quad b_{k\ell}(e_i, e_j) = \begin{cases} 1 & \text{if } (i,j) = (k,\ell) \\ 0 & \text{otherwise.} \end{cases}$

So: $\dim B(V) = (\dim V)^2$ (consistent with $B(V) \simeq \mathrm{Hom}(V, V^*)$!)
The bijection $b \mapsto \varphi_b$ is an isom. of vector spaces !)

* Given a basis $\{e_1 .. e_n\}$ of V, $b : V \times V \to k$ is represented by an $n \times n$ matrix $a_{ij} = b(e_i, e_j)$

$$b\Big(\sum_i x_i e_i, \sum_j y_j e_j\Big) = \sum_{i,j} x_i y_j \, b(e_i, e_j) = (x_1 \dots x_n)\, A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

matrix of b ; $a_{ij} = b(e_i, e_j)$

so: in terms of column vectors, $\underline{\underline{b(X, Y) = X^T A Y.}}$

* Remark: The isomorphism $B(V) \xrightarrow{\sim} \mathrm{Hom}(V, V^*)$ is natural, in the sense that
$b \longmapsto \varphi_b$

SKIP THIS REMARK IF YOUR HEAD HURTS

we have contravariant functors $V \longmapsto B(V)$ and $V \longmapsto \mathrm{Hom}(V, V^*)$,
(on morphisms, $f : V \to W \rightsquigarrow B(f) : B(W) \longrightarrow B(V)$ and $\mathrm{Hom}(W, W^*) \to \mathrm{Hom}(V, V^*)$
$b(\cdot,\cdot) \longmapsto b(f(\cdot), f(\cdot))$ $\quad \varphi \mapsto f^t \circ \varphi \circ f$

and the isom's $B(V) \xrightarrow{\sim} \mathrm{Hom}(V, V^*)$ define a natural transformation between them.