 Reminder: midterm due tonight

Last time: began a digression on categories. A category has objects & morphisms + operation: composition of morphisms.

Def: $\mathcal{C}, \mathcal{D}$ categories. A (covariant) functor $F: \mathcal{C} \to \mathcal{D}$ is an assignment
- to each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
- to each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$, a morphism $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$

s.t. 1) $F(id_X) = id_{F(X)}$ 2) $F(g \circ f) = F(g) \circ F(f)$.

* A contravariant functor = same except direction of morphisms is reversed:

$$f \in \mathrm{Mor}_{\mathcal{C}}(X,Y) \mapsto F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(Y), F(X)) \quad ; \quad F(g \circ f) = F(f) \circ F(g).$$

Ex: on Vect_k, $V \mapsto V^*$ dual (see HW5).

* There's one more layer to this, if you love category theory: given 2 functors $F, G: \mathcal{C} \to \mathcal{D}$, a natural transformation t from F to G is the data, $\forall X \in \mathrm{ob}\,\mathcal{C}$, of a morphism $t_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X))$, s.t. $\forall X, Y \in \mathrm{ob}\,\mathcal{C}$, $\forall f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$,

$$\begin{array}{ccc} F(X) & \xrightarrow{t_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[t_Y]{} & G(Y) \end{array}$$

commutes in $\mathcal{D}$.

Ex: on Vect_k, $V \mapsto V^{**}$ double dual is a (covariant) functor. We've said there is a "natural" map

$$ev_V : V \to V^{**} \qquad \text{(isom. if } \dim < \infty)$$
$$v \mapsto (\ell \mapsto \ell(v))$$

The precise meaning is: ev_V is part of a natural transformation of functors $\mathrm{Vect}_k \to \mathrm{Vect}_k$, from the identity functor to the double dual functor.

Bilinear forms:

Def: A bilinear form on a vector space V over field k is a map $b: V \times V \to k$ that is linear in each variable: $\forall u, v, w \in V$, $\forall \lambda \in k$,

$$\begin{cases} b(\lambda v, w) = b(v, \lambda w) = \lambda b(v,w) \\ b(u+v, w) = b(u,w) + b(v,w) \\ b(u, v+w) = b(u,v) + b(u,w). \end{cases}$$

This is not a linear map $V \times V \to k$ $\quad (b(\lambda(v,w)) = b(\lambda v, \lambda w) = \lambda^2 b(v,w) \neq \lambda b(v,w))$.

Def: We say b is symmetric if $b(v,w) = b(w,v)$ $\forall v, w \in V$

skew-symmetric if $b(v,w) = -b(w,v)$

Ex: • the usual dot product on k^n, $(v,w) \mapsto \sum_{i=1}^{n} v_i w_i$ is symmetric.

• $b: k^2 \times k^2 \to k$, $b((x_1,x_2),(y_1,y_2)) = x_1 y_2 - x_2 y_1$ $\left(= \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right)$ is skew symmetric

★ Given a bilinear map $b: V \times V \to k$, we get a linear map $\varphi_b : V \to V^*$ by defining $\varphi_b(v) = b(v,\cdot) \in V^*$ (maps $w \in V$ to $b(v,w) \in k$).

Conversely, $\varphi: V \to V^*$ determines $b(v,w) = (\varphi(v))(w)$ bilinear form.

This defines a bijection $B(V) \xrightarrow{\sim} \mathrm{Hom}(V,V^*)$.

Def: The <u>rank</u> of $b: V \times V \to k$ is the rank of $\varphi_b : V \to V^*$ $(= \dim \mathrm{Im}\, \varphi_b)$.
If φ_b is an isomorphism, say b is <u>nondegenerate</u>.

★ For a given vector space V, $B(V) = \{\text{bilinear forms } V \times V \to k\}$ is a vector space over k. What is its dimension?

If we choose a basis $\{e_1 \dots e_n\}$ for V, it is enough to specify $b(e_i,e_j)$ $\forall i,j$ in order to determine b: by bilinearity, $b\left(\sum_i x_i e_i, \sum_j y_j e_j\right) = \sum_{i,j} x_i y_j\, b(e_i,e_j)$.

The values of $b(e_i,e_j)$ can be chosen freely - eg. a basis of $B(V)$ is given by $(b_{k\ell})_{\substack{1 \le k \le n \\ 1 \le \ell \le n}}$ $\quad b_{k\ell}(e_i,e_j) = \begin{cases} 1 & \text{if } (i,j) = (k,\ell) \\ 0 & \text{otherwise.} \end{cases}$

So: $\dim B(V) = (\dim V)^2$ (consistent with $B(V) \simeq \mathrm{Hom}(V,V^*)$!
The bijection $b \mapsto \varphi_b$ is an isom. of vector spaces!)

★ Given a basis $\{e_1 .. e_n\}$ of V, $b: V \times V \to k$ is represented by an $n \times n$ <u>matrix</u> $a_{ij} = b(e_i,e_j)$

$$b\left(\sum_i x_i e_i, \sum_j y_j e_j\right) = \sum_{i,j} x_i y_j\, b(e_i,e_j) = (x_1 \cdots x_n)\, A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

matrix of b; $a_{ij} = b(e_i,e_j)$

so: in terms of column vectors, $\underline{\underline{b(X,Y) = X^T A Y}}$.

★ <u>Remark</u>: The isomorphism $\begin{array}{ccc} B(V) & \xrightarrow{\sim} & \mathrm{Hom}(V,V^*) \\ b & \longmapsto & \varphi_b \end{array}$ is natural, in the sense that

SKIP THIS REMARK IF YOUR HEAD HURTS

we have contravariant functors $V \mapsto B(V)$ and $V \mapsto \mathrm{Hom}(V,V^*)$,

(on morphisms, $f: V \to W \rightsquigarrow$ $\begin{array}{ccc} B(f): B(W) & \longrightarrow & B(V) \\ b(\cdot,\cdot) & \longmapsto & b(f(\cdot),f(\cdot)) \end{array}$ and $\begin{array}{ccc} \mathrm{Hom}(W,W^*) & \to & \mathrm{Hom}(V,V^*) \\ \varphi & \mapsto & f^t \circ \varphi \circ f \end{array}$)

and the isom's $B(V) \xrightarrow{\sim} \mathrm{Hom}(V,V^*)$ define a natural transformation between them.

* **Def:** If $S \subset V$ is a subspace of a vector space equipped with a bilinear form $b: V \times V \to k$, we define its <u>orthogonal complement</u>
$S^\perp = \{ v \in V / b(v,w) = 0 \ \forall w \in S \}$. This is a vector subspace.

⚠ This is most intuitive if b is symmetric or skew. Otherwise we have to worry about "left-orthogonal" vs. "right-orthogonal" to S.

* **Lemma:** If b is nondegenerate then $\dim S^\perp = \dim V - \dim S$ (else $\geq$)

Proof: $S^\perp = \mathrm{Ker}\left(\begin{array}{c} V \to S^* \\ v \mapsto \varphi_b(v)_{|S} \end{array} \right)$ composition of $\varphi_b : V \to V^*$ and restriction $r: \begin{array}{c} V^* \twoheadrightarrow S^* \\ \ell \mapsto \ell_{|S} \end{array}$

By rank theorem, $\dim S^\perp = \dim V - \mathrm{rank}(r \circ \varphi_b)$. If b is nondegenerate then φ_b isomorphism and r surjective $\Rightarrow \mathrm{rank}(r \circ \varphi_b) = \dim S^* = \dim S$; in general $\leq$ □

Ex:
- $V = \mathbb{R}^n$ with the standard dot product $b(v,w) = \sum_{i=1}^n v_i w_i$: then $V = S \oplus S^\perp$ the "usual" orthogonal complement because: $S \cap S^\perp = \{0\}$ (see below) and $\dim S + \dim S^\perp = \dim V$.
- but for $b: k^2 \times k^2 \to k$
$$b((x_1,x_2),(y_1,y_2)) = x_1 y_2 - x_2 y_1 \quad \text{(skewsymmetric, nondegenerate)}$$
$S \subset k^2$ 1-dim! subspace spanned by any nonzero vector $v \Rightarrow$ <u>$S^\perp = S$!!</u>
(because $b(v,w) = 0 \Leftrightarrow \det(v,w) = 0 \Leftrightarrow w \in k \cdot v = S$).

<u>Inner product spaces</u>:

Def^n: An <u>inner product space</u> is a vector space V over $\mathbb{R}$ together with a <u>symmetric</u> <u>definite positive</u> bilinear form $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$

<u>Symmetric</u>: $\langle u,v \rangle = \langle v,u \rangle$ <u>Def. positive</u>: $\langle u,u \rangle \geq 0 \ \forall u \in V$, and $\langle u,u \rangle = 0$ iff $u = 0$.

This definition only makes sense over an <u>ordered</u> field so "$\langle u,u \rangle \geq 0$" makes sense. In practice this means $\mathbb{R}$. We can't do this over $\mathbb{C}$. (we'll see a workaround: Hermitian forms)

- Let $\varphi : \begin{array}{c} V \to V^* \\ v \mapsto \langle v, \cdot \rangle \end{array}$ be the linear map corresponding to $\langle \cdot , \cdot \rangle$.

$\langle \cdot , \cdot \rangle$ definite positive $\Rightarrow \varphi$ is injective (since $\forall v \neq 0$, $\varphi(v) \neq 0$! $\varphi(v)(v) > 0$).
$\Rightarrow$ (assuming $\dim V < \infty$) φ is an <u>iso.</u> $V \xrightarrow{\sim} V^*$, ie. $\langle \cdot , \cdot \rangle$ is <u>nondegenerate</u>. (The converse is false: $\langle \cdot , \cdot \rangle$ nondegenerate $\not\Rightarrow$ positive).

Prop: V finite-dim inner product space, $S \subset V$ subspace $\Rightarrow V = S \oplus S^\perp$.

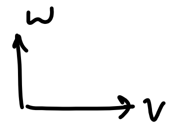
Pf: • we've seen: $\langle \cdot,\cdot \rangle$ is nondegenerate so $\dim S^\perp = \dim V - \dim S$.

• since $\langle \cdot,\cdot \rangle$ is positive definite, $v \in S \cap S^\perp \Rightarrow \langle v, v \rangle = 0 \Rightarrow v = 0$.

So $S \cap S^\perp = \{0\}$. Since dimensions add up to $\dim V$, this implies $S \oplus S^\perp = V$. □

Def: • The <u>norm</u> of a vector is $\|v\| = \sqrt{\langle v, v \rangle}$.

• $v, w \in V$ are <u>orthogonal</u> if $\langle v, w \rangle = 0$.



<u>Observe</u>: $\|v-w\|^2 = \langle v-w, v-w \rangle = \|v\|^2 + \|w\|^2 - 2\langle v, w \rangle$.

→ if v and w are orthogonal then $\|v-w\|^2 = \|v\|^2 + \|w\|^2$ Pythagorean thm

→ in general, by analogy with law of triangles, we define the <u>angle</u> b/w 2 vectors $\measuredangle(v,w) = \cos^{-1}\left(\frac{\langle v, w \rangle}{\|v\|\,\|w\|}\right)$. This only makes sense if $|\langle v, w \rangle| \leq \|v\|\,\|w\|$?

Theorem (<u>Cauchy-Schwarz inequality</u>) $\forall u, v \in V$, $|\langle u, v \rangle| \leq \|u\|\,\|v\|$.

Pf: The inequality is unaffected by scaling so we can assume $\|u\| = 1$.
Decompose v along $V = S \oplus S^\perp$ where $S = \text{span}(u) \subset V$. Explicitly,
$v = v_1 + v_2$, $v_1 = \langle v, u \rangle u \in \text{span}(u)$, $v_2 = v - \langle v, u \rangle u$ orthogonal to u.

Then $v_1 \perp v_2$ so $\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 \geq \|v_1\|^2 = \langle v, u \rangle^2$.

This is the desired inequality for $\|u\| = 1$. □

Def: V finite dim^{l} /$\mathbb{R}$ with inner product $\langle \cdot,\cdot \rangle$. A basis $v_1 \dots v_n$ of V is said to be <u>orthonormal</u> if $\langle v_i, v_j \rangle = \begin{cases} 1 & i = j \quad (\text{ie. } \|v_i\| = 1) \\ 0 & i \neq j \quad (\text{ie. } v_i \perp v_j) \end{cases}$

In such a basis, $(V, \langle \cdot,\cdot \rangle) \cong (\mathbb{R}^n$ with standard dot product$)$.

Thm: Every finite-dimensional inner product space (/$\mathbb{R}$) has an orthonormal basis.

Proof 1: By induction on $\dim(V)$: choose $v \neq 0 \in V$, let $v_1 = \frac{v}{\|v\|}$ so $\|v_1\| = 1$.
Then let $S = \text{span}(v_1)$, $V = S \oplus S^\perp$.
Let $v_2, \dots, v_n$ be an orthonormal basis for $S^\perp$ (the restriction of $\langle \cdot,\cdot \rangle$ to $S^\perp$ is an inner product!)
Then $v_1, \dots v_n$ is an orthonormal basis for V (check!). □

Proof 2: start with any basis $w_1, \dots, w_n$ of V and use the Gram-Schmidt process.
First set $v_1 = \frac{w_1}{\|w_1\|}$. Then take $w_2 - \langle w_2, v_1 \rangle v_1$ which is $\perp v_1$
(and nonzero by independence of w_i), set $v_2 = \frac{w_2 - \langle w_2, v_1 \rangle v_1}{\| w_2 - \langle w_2, v_1 \rangle v_1 \|}$
and so on, set $v_j = \frac{w_j - \sum_{i=1}^{j-1} \langle w_j, v_i \rangle v_i}{\| w_j - \sum_{i=1}^{j-1} \langle w_j, v_i \rangle v_i \|}$. Then $(v_1, \dots, v_n)$ is an orthonormal basis $\square$

So: every finite dim'l inner product space $/\mathbb{R}$ is isomorphic (as an inner product space, not just as a vector space) to standard $\mathbb{R}^n$, $n = \dim V$.