Recall: • bilinear form $b: V\times V \to k$ $\longleftrightarrow$ $\varphi_b \in \mathrm{Hom}(V,V^*)$

$\varphi_b(v) = b(v,\cdot) = \left(\begin{array}{c} V\to k \\ w \mapsto b(v,w)\end{array}\right)$

This gives an isom. $B(V) \simeq \mathrm{Hom}(V,V^*)$.

• b is nondegenerate if φ_b is an isom.

• in a basis $(e_1,\dots,e_n)$, represent b by a matrix A with entries $a_{ij} = b(e_i,e_j)$

$$b(\textstyle\sum x_i e_i, \sum y_j e_j) = \sum a_{ij}\, x_i\, y_j \quad = X^T A Y.$$

b is symmetric iff A is symmetric $(a_{ij} = a_{ji})$

nondegenerate iff A is invertible

• the orthogonal of a subspace $S\subset V$ is $S^\perp = \{v\in V \,/\, b(v,w)=0\ \forall w\in S\}$

If b is non-degenerate, $\dim(S^\perp) = \dim V - \dim S$ (otherwise $\geq$)

but we need not have $S\cap S^\perp = \{0\}$.

Inner product spaces:

Defn: An inner product space is a vector space V over $\mathbb{R}$ together with a symmetric definite positive bilinear form $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{R}$

Symmetric: $\langle u,v\rangle = \langle v,u\rangle$ Def. positive: $\langle u,u\rangle \geq 0\ \forall u\in V$, and $\langle u,u\rangle = 0$ iff $u=0$.

This definition only makes sense over an ordered field so "$\langle u,u\rangle\geq 0$" makes sense. In practice this means $\mathbb{R}$. We can't do this over $\mathbb{C}$. (we'll see a workaround: Hermitian forms)

• Let $\varphi: V\to V^*$, $v\mapsto\langle v,\cdot\rangle$ be the linear map corresponding to $\langle\cdot,\cdot\rangle$.

$\langle\cdot,\cdot\rangle$ definite positive $\Rightarrow$ φ is injective (since $\forall v\neq 0$, $\varphi(v)\neq 0$! $\varphi(v)(v)>0$).

$\Rightarrow$ (assuming $\dim V<\infty$) φ is an iso. $V \xrightarrow{\sim} V^*$, ie. $\langle\cdot,\cdot\rangle$ is nondegenerate. (The converse is false: $\langle\cdot,\cdot\rangle$ nondegenerate $\not\Rightarrow$ positive).

Prop: V finite-dim inner product space, $S\subset V$ subspace $\Rightarrow V = S\oplus S^\perp$.

Pf: • we've seen: $\langle\cdot,\cdot\rangle$ is nondegenerate so $\dim S^\perp = \dim V - \dim S$.

• since $\langle\cdot,\cdot\rangle$ is positive definite, $v\in S\cap S^\perp \Rightarrow \langle v,v\rangle = 0 \Rightarrow v=0$.

So $S\cap S^\perp = \{0\}$. Since dimensions add up to $\dim V$, this implies $S\oplus S^\perp = V$. □

Def: • The norm of a vector is $\|v\| = \sqrt{\langle v,v\rangle}$.

• $v,w\in V$ are orthogonal if $\langle v,w\rangle = 0$.

<u>Observe</u>: $\|v-w\|^2 = \langle v-w, v-w\rangle = \|v\|^2+\|w\|^2 - 2\langle v,w\rangle$.

→ if v and w are orthogonal then $\|v-w\|^2 = \|v\|^2+\|w\|^2$ Pythagorean thm

→ in general, by analogy with law of triangles, we define the <u>angle</u> b/w 2 vectors
$\measuredangle(v,w) = \cos^{-1}\left(\frac{\langle v,w\rangle}{\|v\|\,\|w\|}\right)$. This only makes sense if $|\langle v,w\rangle| \le \|v\|\,\|w\|$?

<u>Theorem</u> (<u>Cauchy-Schwarz inequality</u>) $\forall u,v\in V,\quad |\langle u,v\rangle| \le \|u\|\,\|v\|$.

<u>Pf</u>: The inequality is unaffected by scaling so we can assume $\|u\|=1$.
Decompose v along $V = S\oplus S^\perp$ where $S = \mathrm{span}(u)\subset V$. Explicitly,
$v = v_1+v_2$, $v_1 = \langle v,u\rangle u \in \mathrm{span}(u)$, $v_2 = v - \langle v,u\rangle u$ orthogonal to u.

Then $v_1\perp v_2$ so $\|v\|^2 = \|v_1\|^2+\|v_2\|^2 \ge \|v_1\|^2 = \langle v,u\rangle^2$.

This is the desired inequality for $\|u\|=1$. □

<u>Def</u>: V finite dim! /$\mathbb{R}$ with inner product $\langle\cdot,\cdot\rangle$. A basis $v_1,\dots,v_n$ of V is said to be <u>orthonormal</u> if $\langle v_i,v_j\rangle = \begin{cases}1 & i=j \quad (\text{ie. } \|v_i\|=1)\\ 0 & i\neq j \quad (\text{ie. } v_i\perp v_j)\end{cases}$

In such a basis, $(V,\langle\cdot,\cdot\rangle) \cong$ ($\mathbb{R}^n$ with standard dot product).

<u>Thm</u>: Every finite-dimensional inner product space (/$\mathbb{R}$) has an orthonormal basis.

<u>Proof 1</u>: By induction on $\dim(V)$: choose $v\neq 0\in V$, let $v_1 = \frac{v}{\|v\|}$ so $\|v_1\|=1$.
Then let $S = \mathrm{span}(v_1)$, $V = S\oplus S^\perp$.
Let $v_2,\dots,v_n$ be an orthonormal basis for $S^\perp$ (the restriction of $\langle\cdot,\cdot\rangle$ to $S^\perp$ is an inner product!)
Then $v_1\dots v_n$ is an orthonormal basis for V (check!). □

<u>Proof 2</u>: start with any basis $w_1\dots w_n$ of V and use the <u>Gram-Schmidt process</u>.
First set $v_1 = \frac{w_1}{\|w_1\|}$. Then take $w_2 - \langle w_2,v_1\rangle v_1$ which is $\perp v_1$
(and nonzero by independence of w_i), set $v_2 = \frac{w_2-\langle w_2,v_1\rangle v_1}{\|w_2-\langle w_2,v_1\rangle v_1\|}$

and so on, set $v_j = \dfrac{w_j - \sum_{i=1}^{j-1}\langle w_j,v_i\rangle v_i}{\left\| w_j - \sum_{i=1}^{j-1}\langle w_j,v_i\rangle v_i\right\|}$. Then $(v_1,\dots,v_n)$ is an orthonormal basis □

So: every finite dim! inner product space /$\mathbb{R}$ is isomorphic (as an inner product space, not just as a vector space) to standard $\mathbb{R}^n$, $n = \dim V$.

Operators on inner product spaces: Let $(V, \langle \cdot, \cdot \rangle)$ inner product space. There are two special classes of linear operators on V of interest to us.

Def: Say $T : V \to V$ is an *orthogonal operator* if it respects the inner product, ie. $\langle Tu, Tv \rangle = \langle u, v \rangle \; \forall u, v \in V$.

(In other terms, T "preserves lengths and angles").

Remarks: 1) orthogonal operators map orthonormal bases to orthonormal bases!

$$\langle Te_i, Te_j \rangle = \langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

in particular, orthogonal operators are always invertible!

2) If T is orthogonal then T^{-1} is orthogonal

$$(\langle T^{-1}u, T^{-1}v \rangle \underset{T \text{ orthogonal}}{=} \langle T(T^{-1}u), T(T^{-1}v) \rangle = \langle u, v \rangle \; \forall u, v)$$

If T_1, T_2 are orthogonal then so is $T_1 T_2$ (check!)

Hence: orthogonal operators form a *subgroup of $\mathrm{Aut}(V)$*.

3) If M is the matrix representing T in an orthonormal basis, then $M^T M = I$.

Indeed: entries of $M^T M$ = dot products of columns of M!

$$(M^T M)_{ij} = \sum_k M^T_{ik} M_{kj} = \sum_k M_{ki} M_{kj} = \langle M(e_i), M(e_j) \rangle = \langle e_i, e_j \rangle.$$

Adjoint operator:

Def: Let $T : V \to V$ linear operator on an inner product space. $(V, \langle \cdot, \cdot \rangle)$
There exists a unique linear operator $T^* : V \to V$, called the *adjoint* of φ, such that $\langle v, T(w) \rangle = \langle T^*(v), w \rangle \; \forall v, w \in V$.

Indeed: given $v \in V$, the linear functional
$$V \longrightarrow \mathbb{R}$$
$$w \longmapsto \langle v, T(w) \rangle$$
is, using nondegeneracy of $\langle \cdot, \cdot \rangle$, given by the inner product of w with some element of V, which we call $T^*(v)$; then check this has linear dependence on v.

Equivalently: $\langle\cdot,\cdot\rangle$ defines an isom. $\varphi: V \xrightarrow{\sim} V^*$. Then T^* is the composition of

$$\begin{array}{ccccccc} V & \xrightarrow{\varphi} & V^* & \xrightarrow{T^t} & V^* & \xrightarrow{\varphi^{-1}} & V \\ v & \longmapsto & \langle v,\cdot\rangle & \longmapsto & \langle v, T(\cdot)\rangle = \langle T^*(v),\cdot\rangle & \longmapsto & T^*(v). \end{array}$$

Def: $T: V \to V$ is <u>self-adjoint</u> if $T^* = T$. (ie. $\langle v, Tw\rangle = \langle Tv, w\rangle \ \forall v, w$).

* In an orthonormal basis $(e_1, \dots, e_n)$ of V, $\langle v,w\rangle = V^t\, W$, so
  - transpose gives a row vector
  - column vector

  if matrix of T is M, T^* is N,

$$\left.\begin{array}{l} \langle v, T(w)\rangle = v^t M w \\ \langle T^*(v), w\rangle = (Nv)^t w = v^t N^t w \end{array}\right\} \Rightarrow \text{comparing: } N^t = M, \text{ so } N = M^t.$$

Hence: $\mathcal{M}(T^*) = \mathcal{M}(T)^t$ in orthonormal basis; T is self-adjoint $\Leftrightarrow \mathcal{M}(T)$ <u>symmetric</u>

Note that self-adjoint operators ($\sim$symmetric matrices) need not be invertible.
For example 0 is a self-adjoint operator...

Prop: if T is self-adjoint and $S \subset V$ is an invariant subspace ($T(S) \subset S$) then $S^\perp$ is also an invariant subspace ($T(S^\perp) \subset S^\perp$)

Pf: Let $v \in S^\perp$, then $\forall w \in S$, $T(w) \in S$, so $\langle Tv, w\rangle \underset{(T^*=T)}{=} \langle v, Tw\rangle \underset{(v\in S^\perp,\, Tw\in S)}{=} 0$.
Since $\langle Tv, w\rangle = 0 \ \forall w \in S$, we get: $Tv \in S^\perp$. □

Theorem (the <u>spectral theorem</u> for real self adjoint operators)

If $T: V \to V$ is self adjoint then T is diagonalizable, with real eigenvalues.
Even more, T can be diagonalized in an orthonormal basis of $(V, \langle\cdot,\cdot\rangle)$!

The proof (to be seen next time) uses the following key observation:

Lemma: If T is self-adjoint then $\forall a \in \mathbb{R}_+$, $T^2 + a$ is invertible.

Pf: $\forall v \in V - \{0\}$, $\langle (T^2+a)v, v\rangle = \langle T^2 v, v\rangle + a\langle v,v\rangle$
$= \langle Tv, Tv\rangle + a\langle v,v\rangle = \|Tv\|^2 + a\|v\|^2 > 0$

So $(T^2+a)v \neq 0$. Hence $\ker(T^2+a) = 0$. □

→ **Corollary:** If $p \in \mathbb{R}[x]$ is a quadratic without real roots and $T^* = T$ then $p(T)$ is invertible.

Pf: enough to show $T^2 + bT + c$ is invertible whenever $b^2 - 4c < 0$.
write $T^2 + bT + c = (T + \frac{b}{2})^2 + a$, $a = c - \frac{b^2}{4} > 0$, $T + \frac{b}{2}$ self-adjoint
$\Rightarrow$ by the lemma (applied to $T + \frac{b}{2}$) this is invertible. □