Recall: we're studying operators on an inner product space $(V, \langle\cdot,\cdot\rangle)$
V real vector space (finite-dim.),
$\langle\cdot,\cdot\rangle : V\times V \to \mathbb{R}$ symmetric definite positive bilinear form $\left(\langle u,v\rangle = \langle v,u\rangle,\ \langle u,u\rangle > 0\ \forall u\neq 0\right)$.

Def: Say $T: V\to V$ is an orthogonal operator if $\langle Tu, Tv\rangle = \langle u,v\rangle\ \forall u,v\in V$.

Def: Say T is self-adjoint if $\langle Tu,v\rangle = \langle u, Tv\rangle\ \forall u,v\in V$.

Def: The adjoint of $T: V\to V$ is the unique linear operator $T^*: V\to V$ such that $\langle u, T(v)\rangle = \langle T^*(u), v\rangle\ \forall u,v\in V$.

Prop: (1) $\operatorname{Ker}(T^*) = \operatorname{Im}(T)^\perp$, (2) $\operatorname{Im}(T^*) = \operatorname{Ker}(T)^\perp$

Pf: (1) $T^*u=0 \Leftrightarrow \langle T^*u, v\rangle = 0\ \forall v\in V \Leftrightarrow \langle u, Tv\rangle = 0\ \forall v\in V \Leftrightarrow u\perp \operatorname{Im} T$; similarly for (2). □

* In an orthonormal basis $(e_1,\dots,e_n)$ of V, $\langle v,w\rangle = \underset{\substack{\uparrow\\ \text{transpose gives}\\ \text{a row vector}}}{v^t}\ \underset{\substack{\uparrow\\ \text{column vector}}}{w}$, so
if matrix of T is M, T^* is N,

$$\left.\begin{aligned}\langle v, T(w)\rangle &= v^t M w\\ \langle T^*(v), w\rangle &= (Nv)^t w = v^t N^t w\end{aligned}\right\} \Rightarrow \text{comparing: } N^t = M, \text{ so } N = M^t.$$

Hence: $\mathcal{M}(T^*) = \mathcal{M}(T)^t$ in orthonormal basis; T is self-adjoint $\Leftrightarrow \mathcal{M}(T)$ symmetric

Note that self-adjoint operators ($\sim$symmetric matrices) need not be invertible.
For example 0 is a self-adjoint operator...

Prop: if T is self-adjoint and $S\subset V$ is an invariant subspace ($T(S)\subset S$) then $S^\perp$ is also an invariant subspace ($T(S^\perp)\subset S^\perp$)

Pf: Let $v\in S^\perp$, then $\forall w\in S$, $T(w)\in S$, so $\langle Tv, w\rangle \underset{(T^*=T)}{=} \langle v, Tw\rangle \underset{(v\in S^\perp,\ Tw\in S)}{=} 0$.
Since $\langle Tv, w\rangle = 0\ \forall w\in S$, we get: $Tv\in S^\perp$. □

Lemma: If T is self-adjoint then $\forall a\in\mathbb{R}_+$, T^2+a is invertible.

Pf: $\forall v\in V-\{0\}$, $\langle (T^2+a)v, v\rangle = \langle T^2v, v\rangle + a\langle v,v\rangle$
$= \langle Tv, Tv\rangle + a\langle v,v\rangle = \|Tv\|^2 + a\|v\|^2 > 0$
So $(T^2+a)v\neq 0$. Hence $\operatorname{Ker}(T^2+a) = 0$. □

Corollary: If $p\in\mathbb{R}[x]$ is a quadratic without real roots and $T^*=T$ then $p(T)$ is invertible.

Pf: enough to show T^2+bT+c is invertible whenever $b^2-4c<0$.

write $T^2 + bT + c = (T + \frac{b}{2})^2 + a$, $a = c - \frac{b^2}{4} > 0$, $T + \frac{b}{2}$ self-adjoint
$\Rightarrow$ by the lemma (applied to $T + \frac{b}{2}$) this is invertible. $\square$

$\Rightarrow$ **Theorem** (the spectral theorem for real self adjoint operators)

> If $T: V \to V$ is self adjoint then T is diagonalizable, with real eigenvalues. Even more, T can be diagonalized in an orthonormal basis of $(V, \langle\cdot,\cdot\rangle)$!

Pf:
- First we show the existence of an eigenvector.
  Pick $v \in V$, $v \neq 0$; since $v, Tv, \dots, T^n v \in V$ are linearly dependent ($n = \dim V$), there exists a (nonconstant) polynomial st. $(a_n T^n + \dots + a_0) v = 0$.
  This doesn't factor into degree 1 factors over $\mathbb{R}$ like it would over $\mathbb{C}$, but it factors into linear and quadratic factors

$$\underbrace{\prod (T - \lambda_i)}_{\text{these are the real roots}} \; \underbrace{\prod (T^2 + b_j T + c_j)}_{\text{irreducible (no real roots) coming from pairs of complex conjugate roots.}} v = 0$$

  At least one of these operators must have a nontrivial kernel (else their product is invertible, but $v \mapsto 0$!). By the previous corollary, each $T^2 + b_j T + c_j$ is invertible, so in fact some $T - \lambda_i$ must have a nontrivial kernel, hence an eigenvector!

- Now, diagonalization: we know there's an eigenvector $v_1 \in V$ with eigenvalue $\lambda_1 \in \mathbb{R}$; scaling v_1 if needed we may assume $\|v_1\| = 1$.
  Then $S = \mathrm{span}(v) \subset V$ is an invariant subspace, hence (by Prop above) so is $S^\perp$.
  By induction, using inner product on $S^\perp$ induced by restricting $\langle\cdot,\cdot\rangle$ and observing $T_{|S^\perp}$ is still self-adjoint, there is a basis of $S^\perp$, $(v_2 \dots v_n)$ (orthonormal if we wish), st. each v_j is an eigenvector of T.
  Then $(v_1 \dots v_n)$ is a basis of V in which T diagonalizes, and we can assume it is orthonormal. $\square$

So: $\boxed{T \text{ self adjoint} \rightsquigarrow \mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \text{ in a suitable orthonormal basis.}}$

Rmk: this also implies: eigenvectors of T for distinct eigenvalues are orthogonal! but we already knew this because
$Tv = \lambda v$, $Tw = \mu w \Rightarrow \lambda \langle v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \mu \langle v, w\rangle$,
so $\lambda \neq \mu \Rightarrow v \perp w$.

Next: is there an analogous structure result for orthogonal transformations??

($T: V \to V$ orthogonal $\iff \langle Tu, Tv\rangle = \langle u,v\rangle \ \forall u,v \iff T^* = T^{-1}$)

→ in dim. 1: T is mult. by a scalar, so T orthogonal $\iff T = \pm I$.

→ in dim. 2: T orthogonal $\iff$ T is a rotation or a reflection.

(given orthonormal basis (e_1, e_2), Te_1 is any unit vector $\in$ unit circle $\{v \in V / |v| = 1\} = \{\cos\theta\, e_1 + \sin\theta\, e_2\}$; Te_2 is also unit vector and $\perp Te_1 \Rightarrow 2$ possibilities

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ = rotation by θ.

$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ = reflection

Rotations have no eigenvectors

Reflections have eigenvalues ± 1

two orthogonal eigenspaces

Notation: for $(V, \langle\cdot,\cdot\rangle)$, $SO(V) \subset O(V) \subset GL(V)$ subgroups

$O(V)$: orthogonal; $GL(V)$: invertible linear operators $T: V \to V$

$SO(V)$: subgroup of orientation-preserving orthogonal transformations: those with $\det = +1$.

in dim. 1: $\{+I\}$, in dim 2: rotations

Since $\underset{\langle\cdot\rangle \ \text{std}}{V \simeq \mathbb{R}^n}$ by choosing orthonormal basis, usually write $O(\mathbb{R}^n) = O(n)$

$SO(\mathbb{R}^n) = SO(n)$

$SO(n)$ has index 2 in $O(n)$, $\quad 1 \to SO(n) \to O(n) \xrightarrow[\det]{} \{\pm 1\} \simeq \mathbb{Z}/2 \to 1$.

$SO(2) \simeq S^1$ (rotations $\leftrightarrow$ angles)

Recall: $T: V \to V$ linear operator $\Rightarrow \exists$ invariant subspace $W \subset V$ of dim. 1 or 2

+ if T is orthogonal for $\langle\cdot,\cdot\rangle$ then it maps $W^\perp$ to $(T(W))^\perp = W^\perp$.

⇒ **Thm**: If $T: V \to V$ is an orthogonal operator on a finite dim. inner product space, then V decomposes into a direct sum of orthogonal invariant subspaces

$V = \bigoplus V_i$, $\quad \underset{(\text{ie. } V_i \subset V_j^\perp)}{V_i \perp V_j \ \forall i \neq j}$, $\quad T(V_i) = V_i$, $\quad$ of $\dim V_i \in \{1, 2\}$.

and if $\dim V_i = 1$ then $T_{|V_i} = \pm I$

if $\dim V_i = 2$ then $T_{|V_i}$ is either a rotation or reflection

(in latter case, can further decompose into ± 1 eigenspaces, so can replace reflections by 1-dim blocks)

This gives a very nice way to think about an individual transformation as built from reflections and rotations on individual subspaces, but it's pretty useless for understanding the composition of two orthogonal transformations (whose invariant subspaces are unrelated).

Now on to the analogue of all this for complex vector spaces: Hermitian inner products

As previously noted, a bilinear form on a complex vector space $V \times V \to \mathbb{C}$ can't be definite positive, since $b(iv, iv) = -b(v,v)$. Solution: abandon $\mathbb{C}$-linearity in one of the two variables, and only require "conjugate linear"

Def: A Hermitian form on a complex vector space V is $H: V \times V \to \mathbb{C}$ s.t.
H is sesquilinear:
- $H(u+v, w) = H(u,w) + H(v,w)$, $H(u, v+w) = H(u,v) + H(u,w)$.
- $H(u, \lambda v) = \lambda H(u,v)$, however $H(\lambda u, v) = \bar{\lambda} H(u,v)$

  $\hookrightarrow$ conjugate: $\overline{a+ib} = a - ib$.

$+ H$ conjugate-symmetric: $H(u,v) = \overline{H(v,u)}$.

⚠ This is as in Artin. Axler has $\bar{\lambda}$ for the second input.

Conjugate symmetry $\Rightarrow H(u,u) \in \mathbb{R}$.

Def: A Hermitian inner product is a positive definite (conjugate-symmetric) Hermitian form.
$\hookrightarrow$ ie. $H(u,u) \geq 0 \ \forall u$, $H(u,u) = 0 \Leftrightarrow u = 0$.

Rmk: $\varphi_H : V \to V^*$, $u \mapsto H(u, \cdot)$ is now a complex antilinear map $V \to V^*$! $(\varphi(\lambda u) = \bar{\lambda} \varphi(u))$.

Still, various things carry over from the real case:

- H positive definite $\Rightarrow$ H nondegenerate (ie. $\operatorname{Ker} \varphi_H = 0$)
- Given a subspace $W \subset V$, its orthogonal $W^\perp = \{v \in V \,/\, H(v,w) = 0 \ \forall w \in W\}$ is also a subspace, $V = W \oplus W^\perp$. ($\mathbb{C}$-antilinearity doesn't affect $W^\perp$ being a $\mathbb{C}$-subspace; positive definite implies $W \cap W^\perp = \{0\}$).
- **Def:** An orthonormal basis of V with a Hermitian inner product is a basis $\{e_i\}$ such that $H(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else.} \end{cases}$

Thm: V admits an orthonormal basis

Same proof as in real case (by induction on $\dim V$: first pick v_1 with $\|v_1\|^2 = H(v_1,v_1) = 1$, then take an orthonormal basis $v_2 \dots v_n$ of $\operatorname{span}(v_1)^\perp$) (or Gram-Schmidt ...).

Corollary: Every finite dim. Hermitian inner product space is isomorphic to $\mathbb{C}^n$ with the standard Hermitian inner product, $H(z,w) = \sum_j \bar{z}_j w_j$.

In matrix form: $H(z,w) = z^* w$ where $z^* = \bar{z}^T = (\bar{z}_1 \dots \bar{z}_n)$ conjugate transpose.

<u>Not-quite-example</u> (Fourier series) $V = C^\infty(S^1, \mathbb{C})$ infinitely differentiable functions $S^1 \simeq \mathbb{R}/\mathbb{Z} \to \mathbb{C}$

($\Leftrightarrow$ 1-periodic functions $\mathbb{R} \to \mathbb{C}$)

def. $\langle f, g \rangle = \int_{S^1} \overline{f}(t)\, g(t)\, dt$

then $f_n(t) = e^{2\pi i n t}$ are orthogonal, $\langle f_n, f_m \rangle = \delta_{m,n}$.

$\{f_n\}_{n \in \mathbb{Z}}$ not a basis of V, their span $W \subset V$ = space of trigonometric polynomials.

Can think of Fourier series as orthogonal projection onto W.

(Will make more sense with some analysis ... or even better, Hilbert spaces)

<u>Next:</u> linear operators on Hermitian inner product spaces & the complex spectral theorem!