Today: Hermitian inner products on complex vector spaces:

Def: A Hermitian inner product $H: V\times V \to \mathbb{C}$ is a positive definite (conjugate-symmetric) Hermitian form.

$H(u,u)>0\ \forall u\neq 0$; $H(u,v)=\overline{H(v,u)}$, $H(\lambda u,v)=\bar\lambda H(u,v)$, $H(u,\lambda v)=\lambda H(u,v)$.

$\varphi_H: V\to V^*$, $u\mapsto H(u,\cdot)$ is now a complex antilinear map $V\to V^*$! $(\varphi_H(\lambda u)=\bar\lambda\varphi_H(u))$.

Many things carry over from the real case:

- H positive definite $\Rightarrow$ H nondegenerate (ie. $\ker\varphi_H=0$)
- Given a subspace $W\subset V$, its orthogonal $W^\perp=\{v\in V \,/\, H(v,w)=0\ \forall w\in W\}$ is also a subspace, $V=W\oplus W^\perp$.

($\mathbb{C}$-antilinearity doesn't affect $W^\perp$ being a $\mathbb{C}$-subspace; positive definite $\Rightarrow W\cap W^\perp=\{0\}$).

- **Def:** An orthonormal basis of V with a Hermitian inner product is a basis $\{e_i\}$ such that $H(e_i,e_j)=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{else.}\end{cases}$

Thm: V admits an orthonormal basis

Same proof as in real case (by induction on $\dim V$: first pick v_1 with $\|v_1\|^2=H(v_1,v_1)=1$, then take an orthonormal basis $v_2\dots v_n$ of $\mathrm{span}(v_1)^\perp$) (or Gram-Schmidt ...).

Corollary: Every finite dim. Hermitian inner product space is isomorphic to $\mathbb{C}^n$ with the standard Hermitian inner product, $H(z,w)=\sum_j \bar z_j w_j$.

In matrix form: $H(z,w)=z^* w$ where $z^*=\bar z^T=(\bar z_1 \dots \bar z_n)$ conjugate transpose.

Not-quite-example (Fourier series) $V=C^\infty(S^1,\mathbb{C})$ infinitely differentiable functions $S^1=\mathbb{R}/\mathbb{Z}\to\mathbb{C}$

def. $\langle f,g\rangle=\int_{S^1}\overline{f(t)}\,g(t)\,dt$ ($\Leftrightarrow$ 1-periodic functions $\mathbb{R}\to\mathbb{C}$)

then $f_n(t)=e^{2\pi i n t}$ are orthogonal, $\langle f_n,f_m\rangle=\delta_{m,n}$.

$\{f_n\}_{n\in\mathbb{Z}}$ not a basis of V, their span $W\subset V$ = space of trigonometric polynomials.

Can think of Fourier series as orthogonal projection onto W.
(Will make more sense with some analysis ... or even better, Hilbert spaces)

- **Def:** V complex vect. space, H Hermitian inner product, $T: V \to V$
  - the adjoint of T is $T^*: V \to V$ st. $H(T^*v, w) = H(v, Tw)$ $\forall v, w$
  - T is self-adjoint if $T^* = T$, ($\Leftrightarrow H(Tv, w) = H(v, T^*w)$ $\forall v, w$)
    ie. $H(v, Tw) = H(Tv, w)$ $\forall v, w \in V$
  - T is unitary if $H(Tv, Tw) = H(v, w)$ $\forall v, w \in V$ ie. $T^* = T^{-1}$.
  - T is normal if $TT^* = T^*T$.

- Unitary operators form a subgroup $U(V, H) \subset \mathrm{Aut}(V)$ $(U(n) \subset GL(n, \mathbb{C}))$
  Note $U(1) \simeq S^1$ (multiplication by any complex number of norm 1).
- Note: in an orthonormal basis, $\mathcal{M}(T^*) = \mathcal{M}(T)^* \left(= \overline{\mathcal{M}(T)^t}\right)$.
  This is because $H(Tv, w) = (Mv)^* w = v^* M^* w = H(v, T^*w)$ ✓.
  So: self-adjoint complex operators are described by Hermitian matrices, $a_{ij} = \overline{a_{ji}}$.

The complex spectral theorem:

> V finite-dim'l complex vector space, $H: V \times V \to \mathbb{C}$ Hermitian inner product,
> $T: V \to V$ self-adjoint $(T^* = T)$ or unitary $(T^* = T^{-1})$ or normal $(TT^* = T^*T)$
> $\Rightarrow$ there exists an orthonormal basis consisting of eigenvectors of T,
> ie. T is diagonalizable in an orthonormal basis.
> with eigenvalues $\in \mathbb{R}$ if self-adjoint / $\in S^1$ (unit circle) if unitary.

Proof: • Let $v_1 \in V$ be an eigenvector (exists since $\mathbb{C}$ alg. closed), $Tv_1 = \lambda_1 v_1$, $\|v_1\| = 1$.

Claim 1: v_1 is also an eigenvector of T^*, $T^* v_1 = \overline{\lambda_1} v_1$

Indeed,
$$\begin{aligned}\|(T^* - \bar\lambda)v_1\|^2 &= H((T^* - \bar\lambda)v_1, (T^* - \bar\lambda)v_1) \\ &= H(v_1, (T-\lambda)(T^* - \bar\lambda)v_1) \quad \text{(since } (T-\lambda)^* = T^* - \bar\lambda) \\ &= H(v_1, (T^* - \bar\lambda)\underbrace{(T-\lambda)v_1}_{0}) \quad (T, T^* \text{ commute} \Rightarrow \text{so do } T-\lambda, T^*-\bar\lambda). \\ &= 0\end{aligned}$$

Claim 2: $W = \mathrm{span}(v_1)^\perp$ is invariant under T and T^*.

Indeed, $w \in W \Rightarrow$ $H(Tw, v_1) = H(w, T^*v_1) = \bar\lambda H(w, v_1) = 0 \Rightarrow Tw \in W$.
$H(T^*w, v_1) = H(w, Tv_1) = \lambda H(w, v_1) = 0 \Rightarrow T^*w \in W$.

and $(T_{|W})^* = (T^*)_{|W}$ $(H(w_1, Tw_2) = H(T^*w_1, w_2)$ $\forall w_1, w_2 \in W$ ✓$)$.

$\to$ proof by induction: take $\{v_1\} \cup \{$orthonormal basis of $W = v_1^\perp$ diagonalizing $T_{|W}\}$.

★ Back to (not necess. definite) <u>nondegenerate symmetric bilinear forms</u>:

Suppose V is a finite-dimensional vector space over k and $B: V\times V\to k$ is a nondegenerate symmetric bilinear form. Can we classify such B?

(<u>Rmk</u>: $Q(v)=B(v,v): V\to k$ is something called a <u>quadratic form</u>. can recover B from Q if $\mathrm{char}(k)\neq 2$: $B(u,v)=\frac{1}{2}\big(Q(u+v)-Q(u)-Q(v)\big)$).

<u>Classification approach</u>:

find some vector v st. $B(v,v)\neq 0$, and then look at $\mathrm{span}(v)^\perp$

($\mathrm{span}(v)^\perp=\ker(\varphi_B(v): V\to k)$, so $V=\mathrm{span}(v)\oplus\mathrm{span}(v)^\perp$ when $B(v,v)\neq 0$).

Then study $B|_{\mathrm{span}(v)^\perp}$ ...

⚠ Hermitian forms are what most "normal" people care about, however.

<u>Prop</u>: Over $\mathbb{C}$, any nondegenerate symmetric bilinear form admits a basis $e_1\dots e_n$ st. $B(e_i,e_j)=\delta_{ij}=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases}$

<u>Proof</u>:
- since $B(u,v)\neq 0\Rightarrow$ one of $B(u,u)$, $B(v,v)$, $B(u+v,u+v)$ nonzero, B nonzero implies the existence of v st. $B(v,v)\neq 0$.
- Let $e_1=B(v,v)^{-1/2}v$. Then consider $\mathrm{span}(e_1)^\perp=W$. $\mathrm{span}(e_1)\cap\mathrm{span}(e_1)^\perp=\{0\}$ since $B(e_1,e_1)\neq 0$, and $\dim W=\dim\ker B(e_1,\cdot)=\dim V-1\Rightarrow V=\mathrm{span}(e_1)\oplus W$.
- The restriction of B to W is nondegenerate because the matrix of B in basis $\{e_1$, some basis of $W\}$ is invertible (rank n) iff $B|_W$ invertible (rank $n-1$). $\left(\begin{array}{c|c}1 & 0\\ \hline 0 & B|_W\end{array}\right)$
- Complete the proof by induction on dimension (assuming result holds in dim. $n-1$, take e_1 + basis of W st. $B|_W(e_j,e_k)=\delta_{jk}$). □

<u>Prop</u>: Over $\mathbb{R}$, any nondegenerate symmetric bilinear form admits a basis st. $B(e_i,e_j)=\begin{cases}0 & i\neq j\\ \pm 1 & i=j.\end{cases}$

ie. can assume $B\left(\sum_{i=1}^n x_ie_i,\sum_{i=1}^n y_ie_i\right)=\sum_{i=1}^k x_iy_i-\sum_{i=k+1}^n x_iy_i$. $B=\begin{pmatrix}\overbrace{1 \ \ddots \ 1}^{k} & \\ & \overbrace{-1 \ \ddots \ -1}^{n-k}\end{pmatrix}$

We say B has <u>signature</u> $(k,n-k)$. (Case $(n,0)$ = def. positive).

(k = max dim. of a subspace s.t. $B|_W$ definite positive, $n-k$ = —— " —— def. negative).

<u>Proof</u> same as in complex case, except can't always scale to $B(e_1,e_1)=1$, instead we can only force $B(e_1,e_1)=\pm 1$.

- Over $\mathbb{Q}$, things get <u>much</u> harder – number theory enters!

<u>Ex:</u> $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad Q(v) = B(v,v) = v_1^2 + v_2^2$

$\nexists\, v = (v_1, v_2) \in \mathbb{Q}^2$ st. $B(v,v) = v_1^2 + v_2^2 = 3$

$\hookrightarrow$ clearing denominators, get $n_1^2 + n_2^2 = 3m^2$

$n_1, n_2, m \in \mathbb{Z}$ no common factor (esp not all even)

However $n_1^2 + n_2^2 \equiv 0, 1, 2 \bmod 4$ $\quad 3m^2 \equiv 0, 3 \bmod 4$

$\Rightarrow$ necessarily all are even, contradiction.

whereas $B' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ does have $\exists v$ st. $B'(v,v) = 3$ $\quad (v = (1,1))$

- What about the <u>skew-symmetric</u> case? (suppose $\mathrm{char}(k) \neq 2$)

We can still find a "<u>standard basis</u>" for V finite dim. vect. space with $B : V \times V \to k$ nondegenerate skew-symmetric bilinear form (a.k.a: symplectic form) but the process is slightly different since $B(v,v) = 0 \ \forall v \in V$.

Instead: pick any nonzero $e_1 \in V$; since B is nondegenerate, $B(e_1, \cdot) : V \to k$ is nonzero $\Rightarrow \exists f_1 \in V$ st. $B(e_1, f_1) \neq 0$, can make it $= 1$ by scaling f_1.

Now we find $\mathrm{span}(e_1, f_1) \cap \mathrm{span}(e_1, f_1)^\perp = \{0\}$ (if $v = ae_1 + bf_1$ has $B(v, e_1) = B(v, f_1) = 0 \Rightarrow a = b = 0$)

so $V = \mathrm{span}(e_1, f_1) \oplus \mathrm{span}(e_1, f_1)^\perp$, and study the restriction of B to the latter subspace (induction on dim.).

$\Rightarrow$ <u>Prop:</u> V finite dim! over k, $\mathrm{char}(k) \neq 2$, B nondegenerate skewsymmetric bilinear form $V \times V \to k$ $\Rightarrow \dim V$ is even, and V has a basis $(e_1, f_1, \dots, e_n, f_n)$ st. $B(e_i, e_j) = B(f_i, f_j) = 0$, $\ B(e_i, f_j) = \delta_{ij} = -B(f_j, e_i)$.

ie. matrix of B is
$$\begin{pmatrix} \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} & & \\ & \boxed{\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}} & \\ & & \ddots \end{pmatrix}$$

The group of linear transformations preserving B is called the <u>symplectic group</u> $Sp(V, B) \simeq Sp(2n, k)$.

<u>Next time:</u> tensor product & multilinear algebra. This gives us a way to think of bilinear (or <u>multilinear</u>) maps $V_1 \times V_2 \to W$ as linear maps from a new vector space $V_1 \otimes V_2$.