Tensors and multilinear algebra – see handout.

V, W finite dimensional vector spaces over k $\Rightarrow$ the tensor product is a vector space $V\otimes W$ + a bilinear map $V\times W \to V\otimes W$.
$$(v,w)\mapsto v\otimes w$$

Three definitions (from concrete to abstract; all are equivalent ie. give same output up to natural isomorphism)

- Def. 1: Choose bases $e_1\dots e_m$ of V, $f_1\dots f_n$ of W. Then $V\otimes W$ is the vector space with basis $\{e_i\otimes f_j,\ 1\le i\le m,\ 1\le j\le n\}$.

  The bilinear map is $(e_i, f_j)\mapsto e_i\otimes f_j$ + extend by linearity.

  Elements of the form $v\otimes w = (\sum a_i e_i)\otimes(\sum b_j f_j) = \sum a_i b_j\,(e_i\otimes f_j)$ are called pure tensors; not every element of $V\otimes W$ is of this form!
  The rank of an element of $V\otimes W$ = minimal number of terms needed to express it as a linear combination of pure tensors.

This is concrete & makes it clear that $\dim(V\otimes W) = mn$, but the independence of the choice of basis isn't obvious. To de-emphasize the basis:

- Def 2: Start with a vector space U with basis $\{v\otimes w \mid v\in V, w\in W\}$. (This is insanely large: usually this basis is uncountable!), and quotient it by a subspace R of relations among these elements:
$$R\subset U = \text{the span of } \begin{array}{ll}(\lambda v)\otimes w - \lambda(v\otimes w) & \forall \lambda, v, w\\ v\otimes(\lambda w) - \lambda(v\otimes w) & \\ (u+v)\otimes w - u\otimes w - v\otimes w & \forall u,v,w.\\ u\otimes(v+w) - u\otimes v - u\otimes w & \end{array}$$
  Defining $V\otimes W = U/R$ sets all these to zero, enforcing bilinearity of the map $(v,w)\mapsto v\otimes w$.

This shows independence on the basis, but involves an unpleasantly large construction (at the end, if we have bases $\{e_i\}$ of V, $\{f_j\}$ of W, the relations in R do show all elements of $V\otimes W$ are linear combinations of $e_i\otimes f_j$, but before one checks this it's not even obvious that $\dim(V\otimes W)<\infty$)

- The least concrete, yet most mathematically satisfactory definition, characterizes what $V\otimes W$ does without spelling out how it's actually constructed:

namely, that linear maps from $V \otimes W$ to another space, when evaluated on pure tensors $v \otimes w$, give maps from $V \times W$ that are bilinear in v and w.

(eg. in Def. 2: U is too big, quotient by R enforces bilinearity)

<u>Def 3.</u>: The tensor product $V \otimes W$ is the universal vector space through which all bilinear maps from $V \times W$ factor, ie. it is a vector space $V \otimes W$ + a bilinear map $\beta : V \times W \to V \otimes W$ such that, given any vector space U over k, and any bilinear map $b : V \times W \to U$, there exists a unique linear map $\varphi : V \otimes W \to U$ st. $b = \varphi \circ \beta$

$$\begin{tikzcd} V \times W \arrow[r, "b"] \arrow[d, "\beta"'] & U \\ V \otimes W \arrow[ur, dashed, "\exists! \varphi"'] & \end{tikzcd}$$

This tells us the key property of $V \otimes W$ and implies uniqueness up to isomorphism (the univ. property gives isom's between any two candidate constructions of $V \otimes W$), but existence ultimately comes from one of the previous constructions!

<u>Check</u>: Def. 1 satisfies the property: given bases $\{e_i\}$ & $\{f_j\}$ of V and W,

$$\{\text{bilinear maps } b : V \times W \to U\} \longleftrightarrow \{\text{linear maps } \varphi : V \otimes W \to U\}$$

by defining $b(e_i, f_j) = \varphi(e_i \otimes f_j)$ and vice versa.

<u>Basic properties</u>:

- $\otimes : \mathrm{Vect}_k \times \mathrm{Vect}_k \to \mathrm{Vect}_k$ is a functor. This means: given linear maps $\left\{ \begin{array}{l} f : V \to V' \\ g : W \to W' \end{array} \right\}$ we get a linear map $f \otimes g : V \otimes W \to V' \otimes W'$ on pure elements: $(f \otimes g)(v \otimes w) = f(v) \otimes g(w)$. and this respects composition.
- $V \otimes W \cong W \otimes V$ (natural isom., could even claim they're equal ...)
- $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$

More surprising but extremely useful: $\boxed{\mathrm{Hom}(V, W) \simeq V^* \otimes W}$

<u>Proof</u>: the map

$$\begin{aligned} V^* \times W &\longrightarrow \mathrm{Hom}(V, W) \\ (\ell, w) &\longmapsto (v \mapsto \ell(v) w) \end{aligned}$$

is bilinear

so by univ. property we get a linear map $V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$ which takes $\ell \otimes w \mapsto (v \mapsto \ell(v) w)$.

Pick bases $(e_1 \dots e_n)$ of V, $(f_1 \dots f_m)$ of W, let $(e_1^* \dots e_n^*)$ dual basis of V^*.

Then $(e_i^* \otimes f_j)$ basis of $V^* \otimes W$.

The above construction takes $(e_i^* \otimes f_j)$ to $\varphi_{ij} : V \longrightarrow W$

$$v \longmapsto e_i^*(v)\, f_j$$

whose action on basis vectors is: e_i maps to f_j, all others to 0.

Thus $\mathcal{M}(\varphi_{ij}) =$ $m \times n$ matrix with a single nonzero entry
$$\begin{array}{cc} & i \\ j & \begin{pmatrix} & \vdots & \\ \cdots & 1 & \\ & & \end{pmatrix} \end{array}$$

These form a basis of $\mathrm{Hom}(V,W)$.

Since it maps a basis to a basis, $V^* \otimes W \longrightarrow \mathrm{Hom}(V,W)$ is an isom. □

* <u>Ex:</u> if V has basis (e_1, e_2), V^* (e_1^*, e_2^*), & W has basis f_1, f_2, then the linear map with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $e_1^* \otimes (af_1 + cf_2) + e_2^* \otimes (bf_1 + df_2)$.

This is in general a rank 2 tensor, except if $ad - bc = 0$, then can write it as a pure tensor $(xe_1^* + ye_2^*) \otimes (zf_1 + wf_2)$

<u>Fact:</u> ‖ Tensor rank in $V^* \otimes W$ is the same as rank in $\mathrm{Hom}(V,W)$!
(hence the name).

(Rank 1 case: $\ell \otimes w$ corresponds to $(v \mapsto \ell(v) w)$ whose image $= \mathrm{span}(w)$!)

Easiest to see if take basis of V in which $e_{r+1} \dots e_n$ basis of $\ker \varphi$ and of W in which $f_1 \dots f_r$ basis of $\mathrm{Im}\, \varphi$, with $f_i = \varphi(e_i)\ \forall 1 \leq i \leq r$.

Then φ corresponds to $\sum_{i=1}^{r} e_i^* \otimes f_i$. $\left(\Leftrightarrow \mathcal{M}(\varphi) = \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{r = \mathrm{rank}\, \varphi} & 0 \\ \hline 0 & 0 \end{array}\right)\right)$

The isomorphism $\mathrm{Hom}(V,W) \simeq V^* \otimes W$ also implies:

- $(V \otimes W)^* \simeq V^* \otimes W^*$. Can view this as:

$$\begin{aligned} (V \otimes W)^* = \mathrm{Hom}(V \otimes W, k) &= \{\text{Bilinear maps } V \times W \to k\} \\ &\simeq \mathrm{Hom}(V, W^*) \quad \Big(\text{via } b \mapsto \varphi_b : \underset{\in V}{v} \mapsto \underset{\in W^*}{b(v,\cdot)}\Big) \\ &\simeq V^* \otimes W^* \end{aligned}$$

- $\mathrm{Hom}(V,W) \simeq V^* \otimes W \simeq (W^*)^* \otimes V^* \simeq \mathrm{Hom}(W^*, V^*)$

This is actually the <u>transpose</u> construction $\varphi \in \mathrm{Hom}(V,W) \mapsto \varphi^t : W^* \to V^*$.

(easiest to check on rank 1 $\varphi(v) = \ell(v) w \rightsquigarrow \varphi^t(\alpha) = \alpha \circ \varphi = \alpha(w)\, \ell = ev_w(\alpha)\, \ell$
$\ell \otimes w \rightsquigarrow ev_w \otimes \ell$.)

- We can now properly define the <u>trace of a linear operator</u>!

In "ordinary" linear algebra classes, one defines the trace of an $n \times n$ matrix $A = (a_{ij})$ to be $tr(A) = \sum_{i=1}^{n} a_{ii}$ sum of diagonal entries, then noting that $tr(AB) = \sum_{i,j} a_{ij} b_{ji} = tr(BA)$ we have $tr(P^{-1}AP) = tr(A)$ and so the trace of $T: V \to V$ is defined to be the trace of $\mathcal{M}(T)$ in any basis.

We could also try to define the trace via eigenvalues and their multiplicities, over an alg. closed field: in a basis where $\mathcal{M}(T)$ is triangular it is manifest that $tr(T) = \sum n_i \lambda_i$

- We can do better (conceptually), by using $Hom(V,V) \simeq V^* \otimes V$, and the <u>contraction</u> linear map $V^* \otimes V \to k$. Namely, there's a natural bilinear pairing $ev: V^* \times V \to k$, $(\ell, v) \mapsto \ell(v)$ and it determines $tr: V^* \otimes V \to k$ on pure tensors, $\ell \otimes v \mapsto \ell(v)$

  This is indeed equivalent to the usual def^n: choosing a basis (e_i) and the dual basis (e_i^*), $tr(e_i^* \otimes e_j) = e_i^*(e_j) = \delta_{ij}$ $\leftrightarrow$ trace of the matrix with single entry 1 in pos. (j,i).

<u>Def.</u> A map $m: V_1 \times \dots \times V_k \to W$ is <u>multilinear</u> if it is linear in each variable separately.

The tensor product $V_1 \otimes \dots \otimes V_k$ can be defined as above, either using bases of $V_1 \dots V_k$, or as a quotient of a universal vector space by relations, or via universal property for multilinear maps:

There is a multilinear map $\mu: V_1 \times \dots \times V_k \to V_1 \otimes \dots \otimes V_k$ s.t.
$(v_1, \dots, v_k) \mapsto v_1 \otimes \dots \otimes v_k$

$\forall W$ vector space, $\forall m: V_1 \times \dots \times V_k \to W$ multilinear, $\exists! \varphi \in Hom(V_1 \otimes \dots \otimes V_k, W)$ st. $m = \varphi \circ \mu$

$$\begin{tikzcd} V_1 \times \dots \times V_k \arrow[r, "m"] \arrow[d, "\mu"'] & W \\ V_1 \otimes \dots \otimes V_k \arrow[ur, dashed, "\exists! \varphi"'] & \end{tikzcd}$$

In fact nothing new is happening, because $(U \otimes V) \otimes W = U \otimes (V \otimes W) = U \otimes V \otimes W$.

But ... in the special case of $\underbrace{V \otimes \dots \otimes V}_{n \text{ times}} = V^{\otimes n}$ (by convention $V^{\otimes 0} = k$, $V^{\otimes 1} = V$) we have bilinear maps $V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes (k+\ell)}$ $\forall k, \ell \geq 0$, which taken together define a multiplication on the <u>tensor algebra</u> $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ making it a noncommutative ring.