Math 55a Homework 7

Due Wednesday October 20, 2021.

Material covered: Tensors and multilinear algebra; modules. (Handout; Artin §14.1-14.2)

Tensors: See the lectures and the handout on tensors (available on Canvas under "Files") for a discussion of tensor products and multilinear algebra.

1. Let V and W be vector spaces of dimensions m and n, with $n \geq m$. Show that every element $\phi \in V \otimes W$ has rank at most m; that is, it is expressible as a sum of m or fewer pure tensors. If U, V and W are three finite-dimensional vector spaces, can you bound the rank of elements of the triple tensor product $U \otimes V \otimes W$?

2. Let V be an n-dimensional vector space over a field K. Recall that we have a natural linear *contraction map* $\kappa : V^* \otimes V \to K$ sending $\ell \otimes v \in V^* \otimes V$ to $\ell(v)$.

(a) Show that, under the identification $V^* \otimes V = \mathrm{Hom}(V, V) = \mathrm{End}(V)$, this is simply the *trace*: that is, given a linear map $T : V \to V$, $\kappa(T)$ is the sum of the diagonal entries of any matrix representative of T.

(b) Show that the map $b : \mathrm{End}(V) \times \mathrm{End}(V) \to K$ which sends (S, T) to the trace of the composition $S \circ T$ is bilinear and symmetric. (Try to do this in a "coordinate-free" manner, without introducing matrices)

(c) For which n do there exist endomorphisms S, T of an n-dimensional vector space such that $ST - TS = id$ is the identity?

3. Let $V = \mathbb{R}^n$, and $\mathrm{End}(V) = M_n(\mathbb{R})$ the space of $n \times n$ real matrices. The symmetric bilinear form considered in the previous problem is now explicitly the bilinear map $b : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$ given by $b(M, N) = tr(MN)$.

(a) Show that the symmetric bilinear form b is non-degenerate. (Hint: what is $b(M, M^T)$?)

(b) Find a pair of subspaces $W_\pm \subset M_n(\mathbb{R})$ such that (1) $M_n(\mathbb{R}) = W_+ \oplus W_-$, (2) W_+ and W_- are orthogonal to each other for b, and (3) the restriction b to W_+, resp. W_-, is definite positive, resp. definite negative. What is the *signature* $(\dim W_+, \dim W_-)$ of the symmetric bilinear form b?

4. Let V be an n-dimensional vector space over a field K. What is the dimension of $\mathrm{Sym}^d V$?

5. Let V be a finite dimensional vector space over a field of characteristic zero. Show that there is a natural map $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$. Find the kernel of this map in the case where $\dim V = 2$.

Modules: The following problems deal with *modules over commutative rings*, a generalization of the notion of vector spaces over fields. We will discuss the topic briefly in lecture on October 18 (see also the beginning of Chapter 14 in Artin), but here are the basics, which should suffice without needing to wait for the topic to be discussed in lecture.

Recall that a *commutative ring* R is a set with composition laws $+$ and $\times$ satisfying all the axioms of a field except possibly the existence of multiplicative inverses. A *module* M over the ring R is an abelian group, together with a map $R \times M \to M$ satisfying the usual axioms of scalar multiplication. Homomorphisms of modules over R are defined exactly the same way as linear maps of vector spaces.

Any abelian group $(G, +)$ can be made into a $\mathbb{Z}$-module in exactly one way, by setting $ng = g+\cdots+g$ (n times) for any positive integer n, $(-n)g = -(ng)$, and $0g = 0$. (These identities are forced by the properties of scalar multiplication, for instance distributivity implies $2g = (1+1)g = 1g+1g = g+g$ and so on.) Conversely, a $\mathbb{Z}$-module is an abelian group (forgetting the scalar multiplication), so the two notions are equivalent.

A subset $\Gamma \subset M$ of a module over R is called a *spanning set* (or we just say that Γ *generates* M) if every element of M can be written as a linear combination of elements of Γ; it is said to be *independent* if there are no non-trivial relations of linear dependence among elements of Γ. All modules below are assumed to be finitely generated (i.e., have a finite spanning set).

$R^n = \{(x_1, \dots, x_n) \mid x_i \in R\}$ is a module over R; this is called the *free module of rank n* over R; a finitely generated R-module is called *free* if it is isomorphic to R^n for some n, or equivalently, if it has a basis (i.e., a spanning set whose elements are independent). Most finitely generated modules are not free. For example, $\mathbb{Z}/k$ is a finitely generated $\mathbb{Z}$-module (the single element 1 generates) but it is not free (it is not isomorphic to $\mathbb{Z}^n$ for any n) and does not have a basis (every element x satisfies a non-trivial linear relation, $k\,x = 0$).

6. Define the *direct sum* $M \oplus N$ of two modules over a ring R. Show by giving an example that if $L \subset M$ is a submodule, there need not exist a module N such that $M \cong L \oplus N$, in contrast to the case of finite-dimensional vector spaces over a field.

7. Show by giving an example over the ring $R = k[x, y]$ of polynomials in two variables that, again in contrast to the case of vector spaces over a field, a submodule of a free module need not be free.

8. Let M and N be modules over a ring R. Show that the set of homomorphisms $\phi : M \to N$ can itself be given the structure of an R-module, called $\mathrm{Hom}_R(M, N)$, and describe the following modules:

(a) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^m, \mathbb{Z}^n)$;

(b) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^m, \mathbb{Z}/2)$;

(c) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z})$;

(d) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/3)$;

(e) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/6)$.

9. Given any module M over the ring R, its *dual* is the module $M^* := \mathrm{Hom}_R(M, R)$.

(a) Show that there exists a natural homomorphism $\phi : M \to (M^*)^*$.

(b) Show by example, either over $R = k[x]$ or over $R = \mathbb{Z}$, that (once more in contrast to the case of finite-dimensional vector spaces over a field) the map ϕ need not be an isomorphism, even for finitely generated R-modules.

10. Let R be a commutative ring. Show that R is in fact a field if and only if every finitely generated R-module is free.

(Hint: if $a \neq 0$ is not invertible, consider R/aR, the quotient of R by the subgroup $aR = \{ar,\ r \in R\}$; check that it is an R-module, and study its properties.)

11. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?