Recall: the <u>tensor product</u> = a vector space $V\otimes W$ and a bilinear map $V\times W \to V\otimes W$, $(v,w)\mapsto v\otimes w$

- if $\{e_i\}, \{f_j\}$ bases of V and W, $\{e_i\otimes f_j\}$ basis of $V\otimes W$
- $\{\text{bilinear maps } V\times W\to U\} \simeq \{\text{linear maps } V\otimes W\to U\}$
  $$b(v,w) = \varphi(v\otimes w)$$
- rank of a tensor = minimal # of pure tensors needed to express it as $\sum_{i=1}^{\text{rank}} v_i\otimes w_i$.
- $V^*\otimes W \simeq \mathrm{Hom}(V,W)$
  $\ell\otimes w \mapsto (v\mapsto \ell(v)w)$
  $e_i^*\otimes f_j \mapsto$ linear map whose matrix has 1 in position (j,i), 0 everywhere else.
- $V^*\otimes W^* \simeq (V\otimes W)^* \simeq \{\text{bilinear maps } V\times W\to k\}$.

- $\mathrm{Hom}(V,W) \simeq V^*\otimes W \simeq (W^*)^*\otimes V^* \simeq \mathrm{Hom}(W^*,V^*)$

This is actually the <u>transpose</u> construction $\varphi\in\mathrm{Hom}(V,W)\mapsto \varphi^t : W^*\to V^*$.
(easiest to check on rank 1 $\varphi(v)=\ell(v)w \rightsquigarrow \varphi^t(\alpha)=\alpha\circ\varphi=\alpha(w)\,\ell = ev_w(\alpha)\ell$
$\ell\otimes w \rightsquigarrow ev_w\otimes \ell$.)

- We can now properly define the <u>trace of a linear operator</u>!

In "ordinary" linear algebra classes, one defines the trace of an $n\times n$ matrix $A=(a_{ij})$ to be $tr(A)=\sum_{i=1}^n a_{ii}$ sum of diagonal entries, then noting that $tr(AB)=\sum_{i,j} a_{ij}b_{ji} = tr(BA)$ we have $tr(P^{-1}AP)=tr(A)$ and so the trace of $T:V\to V$ is defined to be the trace of $\mathcal{M}(T)$ in any basis.
We could also try to define the trace via eigenvalues and their multiplicities, over an alg. closed field: in a basis where $\mathcal{M}(T)$ is triangular it is manifest that $tr(T)=\sum n_i\lambda_i$

- We can do better (conceptually), by using $\mathrm{Hom}(V,V)\simeq V^*\otimes V$, and the <u>contraction</u> linear map $V^*\otimes V\to k$. Namely, there's a natural bilinear pairing $ev: V^*\times V\to k$, $(\ell,v)\mapsto \ell(v)$ and it determines $tr: V^*\otimes V\to k$ on pure tensors, $\ell\otimes v\mapsto \ell(v)$

This is indeed equivalent to the usual def^n: choosing a basis (e_i) and the dual basis (e_i^*), $tr(e_i^*\otimes e_j) = e_i^*(e_j)=\delta_{ij}$ $\leftrightarrow$ trace of the matrix with single entry 1 in pos. (j,i).

Def.: $m: V_1 \times \dots \times V_k \to W$ is <u>multilinear</u> if it is linear in each variable separately.

The tensor product $V_1 \otimes \dots \otimes V_k$ can be defined as above, either using bases of $V_1 \dots V_k$, or as a quotient of a universal vector space by relations, or via universal property for multilinear maps:

There is a multilinear map $\mu: V_1 \times \dots \times V_k \to V_1 \otimes \dots \otimes V_k$ s.t.
$$(v_1, \dots, v_k) \mapsto v_1 \otimes \dots \otimes v_k$$
$\forall W$ vector space, $\forall\, m: V_1 \times \dots \times V_k \to W$ multilinear, $\exists!\, \varphi \in \mathrm{Hom}(V_1 \otimes \dots \otimes V_k, W)$ st. $m = \varphi \circ \mu$

$$\begin{tikzcd} V_1 \times \dots \times V_k \arrow[r, "m"] \arrow[d, "\mu"'] & W \\ V_1 \otimes \dots \otimes V_k \arrow[ur, dotted, "\exists!\varphi"'] & \end{tikzcd}$$

In fact nothing new is happening, because $(U \otimes V) \otimes W = U \otimes (V \otimes W) = U \otimes V \otimes W$.

But ... in the special case of $\underbrace{V \otimes \dots \otimes V}_{n \text{ times}} = V^{\otimes n}$ (by convention $V^{\otimes 0} = k$, $V^{\otimes 1} = V$)

we have bilinear maps $V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes (k+\ell)}$ $\forall k, \ell \geq 0$, which taken together define a multiplication on the <u>tensor algebra</u> $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ making it a noncommutative ring.

<u>Symmetric algebra:</u>

Remember: we've seen the space of bilinear forms $B(V) \simeq V^* \otimes V^*$ decomposes into $B(V) = B_{symm} \oplus B_{skew}$ (symmetric & skew-symm. bilinear forms).

Equivalently: there is an <u>involution</u> $\varphi: B(V) \to B(V)$ taking $b(x,y) \mapsto b(y,x)$ or on $V^* \otimes V^*$: $\ell \otimes \ell' \mapsto \ell' \otimes \ell$. $\longrightarrow$ = automorphism s.t. $\varphi^2 = id$.

φ has eigenvalues ± 1 and eigenspaces $\mathrm{Ker}(\varphi - I) = B_{symm}$, $\ker(\varphi + I) = B_{skew}$.

We can also do the same on higher tensor powers of V or V^* (the latter = multilinear forms).

There is an <u>action</u> of the symmetric group S_d on $V^{\otimes d}$,

ie each permutation $\sigma \in S_d$ defines a linear map $V^{\otimes d} \xrightarrow{\sigma} V^{\otimes d}$
$$v_1 \otimes \dots \otimes v_d \mapsto v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}$$
+ this defines a group homomorphism $S_d \to \mathrm{Aut}(V^{\otimes d})$

* **Definition:** A tensor $\eta \in V^{\otimes d}$ is <u>symmetric</u> if $\sigma \cdot \eta = \eta$ $\forall \sigma \in S_d$.
$\mathrm{Sym}^d(V) := \{\text{symmetric tensors}\} \subset V^{\otimes d}$ subspace.

eg. $\mathrm{Sym}^d(V^*) = \{\text{symmetric multilinear forms } m: V \times \dots \times V \to k\}$

ie. $m(v_{\sigma(1)}, \dots, v_{\sigma(d)}) = m(v_1, \dots, v_d)$

If $\mathrm{char}(k)=0$, the symmetric part of a tensor can be determined by averaging: $\alpha : V^{\otimes d} \longrightarrow \mathrm{Sym}^d V$ linear

on pure tensors, $\alpha(v_1 \otimes \dots \otimes v_d) = \frac{1}{d!} \sum_{\sigma \in S_d} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}$.

* Still assuming $\mathrm{char}(k)=0$, we could instead define $\mathrm{Sym}^d(V)$ as the <u>quotient</u> of $V^{\otimes d}$ by the subspace spanned by elements of the form $\eta - \sigma(\eta)$, $\eta \in V^{\otimes d}$, $\sigma \in S_d$

explicitly $v_1 \otimes v_2 \otimes v_3 \otimes \dots \otimes v_d - v_2 \otimes v_1 \otimes v_3 \otimes \dots \otimes v_d$ & same for swapping other factors.
(since transpositions generate S_d)

This is different from (but isomorphic to) the previous definition

* to settle the question of which definition (as quotient vs. subspace of $V^{\otimes d}$) is better: the best def^{n} is again by a universal property.

Recall $V^{\otimes d}$ comes with a multilinear map $\mu : V^d \to V^{\otimes d}$ and is characterized by:

$$\mathrm{Hom}(V^{\otimes d}, U) \simeq \{\text{multilinear maps } V^d \to U\} \quad \text{using } \varphi \longmapsto \varphi \circ \mu$$

Now $\mathrm{Sym}^d V$ comes with a <u>symmetric</u> multilinear map $V^d \to \mathrm{Sym}^d V$ and is characterized by:

$$\mathrm{Hom}(\mathrm{Sym}^d V, U) \simeq \{\underline{\text{symmetric}} \text{ multilinear } V^d \to U\}.$$

* The product operations $V^{\otimes k} \times V^{\otimes l} \longrightarrow V^{\otimes k+l}$ induce a product $\mathrm{Sym}^k V \times \mathrm{Sym}^l V \to \mathrm{Sym}^{k+l} V$ (using $\otimes$ followed by averaging α).

These combine to a product operation on $\mathrm{Sym}^\bullet(V) := \bigoplus_{d \geq 0} \mathrm{Sym}^d(V)$, called the <u>symmetric algebra</u> of V.

$\mathrm{Sym}^\bullet(V)$ is a <u>commutative ring</u> (+ vector space over k: a <u>k-algebra</u>)

(<u>check</u>: product is still associative despite symmetrization by averaging:
$$\alpha(\alpha(u \otimes v) \otimes w) = \alpha(u \otimes \alpha(v \otimes w)) = \alpha(u \otimes v \otimes w))$$

<u>Concretely</u>: if $e_1 \dots e_n$ basis of V, then $\mathrm{Sym}^\bullet(V) \simeq k[e_1 \dots e_n]$
polynomial expressions in formal variables $e_1 \dots e_n$.

(simply: denoting $\alpha(e_{i_1} \otimes \dots \otimes e_{i_k})$ by $e_{i_1} \cdots e_{i_k}$ and considering finite linear combinations of all these)

- More explicitly: if $e_1 \dots e_n$ basis of V, then any linear form on V, $\ell \in V^*$, is of the form $v = \sum x_i e_i \longmapsto \ell(v) = \sum a_i x_i$ deg 1 polynomial

Symmetric multilinear forms $\eta \in \mathrm{Sym}^d V^*$ are, likewise, polynomials (with only degree d terms): $v = \sum x_i e_i \mapsto \eta(v,\dots,v) = \sum_{i_1\dots i_d} a_{i_1\dots i_d} x_{i_1}\cdots x_{i_d}$.

So: $\mathrm{Sym}^\bullet(V^*) \simeq k[x_1,\dots,x_n]$ polynomials in n variables (where, by a sleight of hand, x_i denotes the i^{th} coordinate of a vector in V as a linear (degree 1 polynomial) function on V, ie. really this is another name for $e_i^* \in V^*$).

Exterior algebra: do the same thing for skew-symmetric, aka alternating, multilinear forms.

Def: $\eta \in V^{\otimes d}$ is alternating if $\sigma(\eta) = (-1)^\sigma \eta \quad \forall \sigma \in S_d$.

$\Lambda^d(V) = \{\text{alternating tensors}\} \subset V^{\otimes d}$.

sign of σ :: -1 for transpositions & products of odd # of them.

- In characteristic zero, we can view $\Lambda^d(V)$ as the image of skew-symmetrization operator $\beta: V^{\otimes d} \longrightarrow \Lambda^d(V)$

$$\beta(v_1 \otimes \dots \otimes v_d) = \frac{1}{d!} \sum_{\sigma \in S_d} (-1)^\sigma v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)} \overset{\text{def}^{\text{n}}/\text{notation}:}{=:} v_1 \wedge \dots \wedge v_d.$$

This is zero whenever $v_i = v_j$ for some $i \neq j$ ... and so by multilinearity, whenever $v_1 \dots v_d$ are linearly dependent. Thus $\Lambda^d(V) = 0$ whenever $d > \dim V$!

- Alternative definitions: $\Lambda^d(V)$ = quotient of $V^{\otimes d}$ by the subspace spanned by $v_1 \otimes v_2 \otimes v_3 \otimes \dots \otimes v_d + v_2 \otimes v_1 \otimes v_3 \otimes \dots \otimes v_d$ and similarly for swapping any two factors

Or: $\Lambda^d(V)$ vector space with an alternating multilinear map $V \times \dots \times V \longrightarrow \Lambda^d V$, $(v_1,\dots,v_d) \longmapsto v_1 \wedge \dots \wedge v_d$ $\quad (v_1 \wedge v_2 = -v_2 \wedge v_1$ etc.$)$

and universal for alternating multilinear maps on $V \times \dots \times V$.

- If $(e_1,\dots,e_n)$ are a basis of V then $e_{i_1} \wedge \dots \wedge e_{i_d}$, $i_1 < \dots < i_d$ basis of $\Lambda^k V$.
- We have a product $\Lambda^k V \times \Lambda^l V \longrightarrow \Lambda^{k+l} V$ induced by tensor algebra + skew symmetrization. $(v_1 \wedge \dots \wedge v_k) \wedge (w_1 \wedge \dots \wedge w_l) = v_1 \wedge \dots \wedge v_k \wedge w_1 \wedge \dots \wedge w_l$.

This makes the exterior algebra $\Lambda^\bullet V = \bigoplus_{d \geq 0} \Lambda^d V$ into a (skew-commutative) ring ie. if $\eta \in \Lambda^k V$, $\xi \in \Lambda^l V$ then $\eta \wedge \xi = (-1)^{kl} \xi \wedge \eta$. (check: $\dim \Lambda^\bullet V = 2^{\dim V}$)