Math 55a Homework 9

Due Wednesday November 3, 2021.

Material covered: Conjugacy classes; class equation; p-groups; the symmetric group. (Artin §7.1-7.5)

1. Let p be a prime number, and let G be any group of order p^3.

(a) What are the possible orders of the center Z of G?

(b) Assume G is not abelian, and let $g \in G$ be an element not in the center Z. What can be the order of its centralizer? (Recall that the centralizer of g is the subgroup consisting of all elements of G that commute with g.)

(c) What are the possible class equations for G?

2. Consider the *Heisenberg group*

$$H = H(3, \mathbb{F}_p) \ := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \right\}.$$

(a) Find the commutator subgroup H' of H (recall that H' is the subgroup generated by all commutators $[g, h] = ghg^{-1}h^{-1}$). What is the quotient H/H'?

(b) Describe all the conjugacy classes in H.

(c) Find all the normal subgroups of H.

3. Let G be the group of affine transformations of $\mathbb{F}_p$ (p prime), i.e. maps $f_{a,b} : \mathbb{F}_p \to \mathbb{F}_p$ of the form $f_{a,b} : x \mapsto ax + b$ for $a, b \in \mathbb{F}_p$, $a \neq 0$.

(a) Find the commutator subgroup G' of G, and describe the quotient G/G'.

(b) Describe all the conjugacy classes in G.

(c) Show that the classification of normal subgroups of G is determined by that of subgroups of $\mathbb{F}_p^\times = (\mathbb{F}_p - \{0\}, \times)$.

(Optional: what about non-normal subgroups of G?)

4. Find all finite groups G that have at most 3 conjugacy classes.

5. Let $\sigma, \tau \in S_n$ be any two permutations. Show that, even though the products $\sigma\tau$ and $\tau\sigma$ may not be equal, they have the same cycle lengths.

6. (a) List the conjugacy classes in the alternating group A_6, and find the number of elements in each.

(b) Use this to prove that A_6 is simple.

7. For what integers n does there exist a surjective homomorphism $\phi : S_n \to S_{n-1}$?

8. Show that a nonabelian group of order 21 exists by finding one explicitly as a subgroup of S_7.

(Note: The shortest way to solve this problem is to observe that one of the groups you have recently encountered contains a subgroup of order 21, and acts on a set with 7 elements. An alternative, more systematic approach is as follows. Sylow's theorems imply that a group of order 21 contains a unique subgroup of order 7. Taking this for granted, you can try to first build an example "by hand", denoting by x an element of order 7, by y an element not in the subgroup generated by x, and figuring out first the order of y, then what yxy^{-1} might be. You can then turn your example into a subgroup of S_7 by finding suitable permutations that x and y might map to.)

9*. (Optional, extra credit) Let $PGL_2(\mathbb{F}_p)$ be the quotient of $GL_2(\mathbb{F}_p)$ (the group of 2×2 invertible matrices with entries in $\mathbb{F}_p$) by the normal subgroup consisting of scalar multiples of the identity.

(a) What is the order of $PGL_2(\mathbb{F}_p)$? Show that $PGL_2(\mathbb{F}_p)$ acts on the set of 1-dimensional subspaces of $(\mathbb{F}_p)^2$, and that this determines a homomorphism $\psi : PGL_2(\mathbb{F}_p) \to S_{p+1}$; what can you say about this homomorphism for $p = 2$ and $p = 3$? (Cf. HW 8 Problem 8).

(b) We now focus on $p = 5$. Show that $\psi : PGL_2(\mathbb{F}_5) \to S_6$ is an injective homomorphism, whose image $H \subset S_6$ acts transitively on $\{1, \dots, 6\}$.

(c) Show that the action of S_6 on the set of left cosets of H (by left multiplication) gives rise a homomorphism $f : S_6 \to S_6$, and that f is an isomorphism from S_6 to itself. (Hint: what can you say about $\mathrm{Ker}(f)$?) Also show that $f(H) \subset S_6$ is contained in a subgroup $S_5 \subset S_6$ of permutations which fix one element of $\{1, \dots, 6\}$.

(d) Comparing H and $f(H)$, show that the automorphism f of S_6 is not an *inner automorphism*, i.e. not a conjugation $c_g : x \mapsto gxg^{-1}$ for some $g \in S_6$.

(This is the only instance of an automorphism of S_n not given by conjugation! For $n \neq 6$ all automorphisms of S_n are inner, and for $n \notin \{2, 6\}$, $Aut(S_n)$ is isomorphic to S_n via $g \mapsto c_g$.)

10. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?