Math 55a Homework 10

Due Wednesday November 10, 2021.

Material covered: Sylow theorems; finitely presented groups. (Artin §7.6-7.10)

1. Let p and q be distinct primes. Show that a group of order pq or p^2q cannot be simple.

2. How many groups of order 33 are there, up to isomorphism?

3*. How many groups of order 18 are there, up to isomorphism?

4*. Prove that, if $n = pq$ is a product of primes such that $p|(q-1)$, then there exists a *unique* non-abelian group of order n up to isomorphism.

(Hint: at some point in the proof, you may end up studying the roots of the polynomial $x^p - 1 \in \mathbb{F}_q[x]$. These include all the elements of order p in the multiplicative group $\mathbb{F}_q^\times = (\mathbb{F}_q - \{0\}, \times)$. How many are there, and how do they relate to each other?)

5. The dihedral group D_n of symmetries of a regular n-gon is generated by a pair of reflections s, t whose axes make an angle of π/n.

(a) Describe the Cayley graph of D_n with respect to the generators $\{s, t\}$. How are the various types of elements of D_n expressed by words in terms of s and t?

(Recall: given a group G with a finite set of generators $\Gamma \subset G$, the Cayley graph has one vertex for each element of G, and $g, g' \in G$ are connected by an edge whenever $g' = g\gamma$ for some $\gamma \in \Gamma$.)

(b) Let G be the quotient of the free group on two generators σ, τ by the smallest normal subgroup containing σ^2 and τ^2. Describe all elements of G, by giving a list of words in the generators σ and τ among which every element of G appears exactly once.

(c) Show that the homomorphism from the free group to D_n which maps σ to s and τ to t factors through the quotient G, and describe the kernel of the resulting homomorphism from G to D_n. Finally, use this to give a presentation of D_n with generators s and t.

(d) What would the Cayley graph and the presentation of D_n be if instead we used as generators the reflection s and the rotation r by angle $2\pi/n$?

6. Consider the matrices $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{Z})$.

(a) Show that A, B, C generate the Heisenberg group $H := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, and give a set of words in the generators A, B, C (and their inverses) among which every element of H appears exactly once. (In other terms, describe a "normal form" expression for every element of H).

(b) Give a presentation of H in terms of the generators A, B, C. Show that your relations describe H, rather than a larger group of which H is a quotient, by checking that any word in A, B, C (and their inverses) reduces to one of the words you gave in part (a).

(c) Show that H has polynomial growth rate, i.e. that the number of elements described by arbitrary words of length at most N in A, B, C and their inverses is bounded between two polynomials in N.

(d) The number of elements of H described by words of length at most N can in fact be bounded between two polynomials of the *same degree* d (with positive leading coefficients). What is the value of d?

(e) How does your answer to the previous question change if we choose a different (finite) set of generators of H?

7. Let $G = \langle x, y \,|\, x^2 = y^3 = (xy)^3 = 1 \rangle$.

(a) Show that every element of G can be expressed as y^k or $y^k x y^\ell$ for some $k, \ell \in \mathbb{Z}/3$. (Hint: what is xyx equal to?)

(b) Construct a homomorphism from G to the alternating group A_4, and use it to show that $G \simeq A_4$. (Note: this implies that the expressions considered in (a) are all distinct!)

8*. (optional, extra credit) Consider the group $SL_2(\mathbb{F}_3)$ of 2×2 matrices with entries in $\mathbb{F}_3$ and determinant 1, and its quotient $PSL_2(\mathbb{F}_3) = SL_2(\mathbb{F}_3)/\{\pm I\}$ by the normal subgroup $\{\pm I\}$. There is a natural homomorphism φ from $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$ to $PSL_2(\mathbb{F}_3)$ given by reducing coefficients mod 3. The kernel of φ is called the *congruence subgroup* $\Gamma(3) \subset PSL_2(\mathbb{Z})$.

Show that the congruence subgroup $\Gamma(3)$ is normally generated by

$$M = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},$$

i.e. that it is the smallest normal subgroup of $PSL_2(\mathbb{Z})$ which contains M.

(Hint: Recall from lecture that $PSL_2(\mathbb{Z})$ has a presentation with generators $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and relations $S^2 = R^3 = 1$. First show that $PSL_2(\mathbb{F}_3) \simeq A_4$, e.g. by considering the action on subspaces of $(\mathbb{F}_3)^2$ as in HW8 and HW9; then use the result of Problem 7).

9. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?