Recommended text: Fulton & Harris, "Rep theory: A first course", part I

Representation theory = the study of group actions on vector spaces, ie. homomorphisms $G \to GL(V)$. (for us, mostly over $k=\mathbb{C}$).

Historically, groups first arose as geometric symmetries, so in the 19^{th} century groups were mostly thought of as subgroups of $GL(n)$, rather than abstract groups! The more modern viewpoint, rather, splits this into the study of groups on their own (what we've studied) + how to think of an abstract group G as a subgroup of $GL(n)$ (what we'll see now).

We'll focus on representations of finite groups, but the problem is also interesting for discrete infinite groups (eg. $SL_2(\mathbb{Z})$, braid groups, ...), or continuous ones (Lie groups: S^1, $SO(3)$, ...)

Def: A representation of a group G is a vector space V (usually finite-dim.; mostly consider $k=\mathbb{C}$) + an action of G on V by linear operators: ie. $G\times V \to V$ st. $\forall g\in G$, $g: V\to V$ linear map.

Equivalently: a homomorphism $\rho: G \to GL(V)$ the group of invertible linear operators $V\to V$.

Def:
- A subrepresentation is a subspace $W\subset V$ which is invariant under G, ie. $gW=W \quad \forall g\in G$.
- A representation is irreducible if it has no nontrivial subrepresentations.

Ex: If $G\simeq \mathbb{Z}/n$ is a cyclic group then a representation of G is a vector space V together with $\varphi=\rho(1): V\to V$ st. $\varphi^n = id_V$. Return briefly to linear algebra:

Lemma: V finite dim. $\mathbb{C}$-vector space, $\varphi: V\to V$ of finite order $\varphi^n=id$ $\Rightarrow \varphi$ is diagonalizable.

Pf: This is because the minimal polynomial of φ divides φ^n-1 hence has simple roots. Explicitly: over $\mathbb{C}$, $\varphi^n-1=0$ factors as $\prod_k(\varphi-\lambda_k)=0$ where $\lambda_k=e^{2\pi i k/n}$.

So the eigenvalues of φ are n^{th} roots of unity ($\varphi(v)=\lambda v \Rightarrow v=\varphi^n(v)=\lambda^n v$), and the generalized eigenspaces $V_{\lambda_k}=\ker(\varphi-\lambda_k)^N$ ($N>\dim V$) give $V=\bigoplus V_{\lambda_k}$ decomposition of V into invariant subspaces of φ.

Since $\prod_{j\neq k}(\varphi-\lambda_j)$ is invertible on V_{λ_k}, we have $(\varphi-\lambda_k)_{|V_{\lambda_k}}=0$, ie. $\varphi_{|V_{\lambda_k}}=\lambda_k\, id$.

Hence φ is diagonalizable. $\square$

Returning to $G=\mathbb{Z}/n$, invariant subspaces of $\varphi=\rho(1)$ are subrepresentations, and

V splits into a direct sum of 1-dimensional (irreducible) representations, $V_i = \mathrm{span}(e_i)$ for e_i basis of eigenvectors of φ.
each given by a homomorphism $\mathbb{Z}/n \to \mathbb{C}^* = GL_1(\mathbb{C})$. (n such).
$$1 \mapsto \lambda = e^{2\pi i k/n}$$

* Now, if V is a $\mathbb{C}$ representation of a finite abelian group G, $\rho: G \to GL(V)$, $G \simeq \mathbb{Z}/m_1 \times \dots \times \mathbb{Z}/m_r$, the G-action is equivalent to the data of $\varphi_1, \dots, \varphi_r : V \to V$ st. $\varphi_i^{m_i} = id_V$, and which pairwise commute $\varphi_i \varphi_j = \varphi_j \varphi_i$.
(then $\sum a_i e_i \mapsto \prod \varphi_i^{a_i}$).

  By the lemma each φ_i is diagonalizable, and by HW, commuting diagonalizable operators are <u>simultaneously</u> diagonalizable. In fact: the eigenspaces of φ_i are invariant under all φ_j, and the restriction of φ_j to an eigenspace of φ_i is of finite order hence diagonalizable by the lemma. Proceed by induction on r.
  This shows that V splits into a $\oplus$ of 1-dimensional subrepresentations
  Those now correspond to homomorphisms $G \to GL_1(\mathbb{C}) \simeq \mathbb{C}^*$.

* for G a finite abelian group, define its <u>dual</u> $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$.
  This is an abelian group using pointwise multiplication:
  if $\rho, \rho' : G \to \mathbb{C}^*$ homomorphisms, then so is $\rho\rho' : G \to \mathbb{C}^*$
  $$g \mapsto \rho(g)\rho'(g).$$
  (⚠ this uses the fact that $\mathbb{C}^*$ is abelian
  $\Rightarrow (\rho\rho')(g_1 g_2) = (\rho\rho')(g_1)(\rho\rho')(g_2)$)

  Concretely, for $G = \mathbb{Z}/n$, $\hat{G} \cong \mathbb{Z}/n$ as well. though there is no canonical map $G \to \hat{G}$.
  $$\rho \mapsto \rho(1) \in \{e^{2\pi i k/n}\} \simeq \mathbb{Z}/n$$

  Similarly, $G = \mathbb{Z}/m_1 \times \dots \times \mathbb{Z}/m_r \Rightarrow \hat{G} \simeq$ same
  (ρ is determined by images of generators of G, which are roots of 1 in $\mathbb{C}^*$)

This completes the classification of (complex) representations of finite abelian groups!

<u>Def</u>: Given two representations V, W of G, a <u>homomorphism of representations</u> $\varphi: V \to W$ is a linear map $\varphi: V \to W$ that is <u>equivariant</u>, ie. compatible with the group actions:
$$\varphi(gv) = g\varphi(v) \ \forall v \in V \ \forall g \in G.$$

We denote the set of homomorphisms of representations (G-equivariant linear maps) by $\mathrm{Hom}_G(V, W)$ (as opposed to all linear maps $\mathrm{Hom}(V, W)$).

We can make new representations out of old ones: in particular:

- If V, W are two rep^s of G and $\varphi \in \mathrm{Hom}_G(V,W)$, then $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$ are preserved by G, hence subrepresentations of V and W.
  ($v \in \ker\varphi \Rightarrow \varphi(gv) = g\varphi(v) = g.0 = 0$ so $gv \in \ker\varphi$).
- If $W \subset V$ is a subrepresentation, then V/W is also a representation.
  (since $g(W) = W$, $g \in G$ maps cosets to cosets: $g(v+W) = gv + W$)
- V, W reps. of $G \Rightarrow V \oplus W$ is also a representation $(g(v,w) = (gv, gw))$
  and so is $V \otimes W$ $(g(v\otimes w) = gv \otimes gw$ + extend by linearity).
- $\mathrm{Hom}(V,W)$ (all linear maps) is also a G-representation, but this requires care: given $\varphi : V \to W$, what can we expect of $g(\varphi): V \to W$?
  Ans: $g(\varphi)(gv) = gw$. So: $g(\varphi) = g \circ \varphi \circ g^{-1} \in \mathrm{Hom}(V,W)$.
  (Check: $(gh)(\varphi) = g(h(\varphi))$ ✓)
  $\uparrow$ action of g on W $\quad$ $\uparrow$ action of g^{-1} on V
  Comparing with the above: given $\varphi \in \mathrm{Hom}(V,W)$,
  $\varphi \in \mathrm{Hom}_G(V,W)$ G-equivariant $\Longleftrightarrow g(\varphi) = \varphi \ \forall g \in G$.
- Specializing to $V^* = \mathrm{Hom}(V,k)$, where k can be equipped with trivial representation ($\forall g \in G$ acts by id): the dual rep^n of V is
  V^* with $g(\ell) = \ell \circ g^{-1}$, ie. g acts on V^* by ${}^t(g^{-1})$
  $\downarrow$ action of g^{-1} on V.
  Then the isom. $V^* \otimes W \simeq \mathrm{Hom}(V,W)$ is an isom of representations (ie. a G-equivariant isom)
  $(g(\ell \otimes w) = (\ell\circ g^{-1}) \otimes gw$ does map $v \mapsto \ell(g^{-1}v)\, gw$).

Theorem. Let V be any rep. of a finite group G (over $\mathbb{C}$, or k of char. 0), and suppose $W \subset V$ is an invariant subspace (ie., subrepresentation). Then there exists another invariant subspace $U \subset V$ st. $V = U \oplus W$.
(as a direct sum of rep^{n}s)

Corollary: any finite dim. representation of a finite gp decomposes into direct sum of irreducibles.

Two proofs of thm. The first one uses:

Lemma: If V is a $\mathbb{C}$-representation of a finite group G, then there exists a positive definite Hermitian inner product on V which is preserved by G: $H(gv, gw) = H(v,w)$ $\forall g, v, w$, ie. all the linear operators $g: V \to V$ are <u>unitary</u>.

Pf. lemma: Let H_0 be any Hermitian inner product on V, and use averaging trick to set

$$H(v,w) = \frac{1}{|G|}\sum_{g\in G} H_0(gv, gw).$$

Then H is still Hermitian and definite positive (hence an inner product), and $H(gv, gw) = H(v,w)$. □

Pf. thm: Equip V with a G-invariant Hermitian inner product H as in the Lemma. Then if $g(W)=W$, g unitary $\Rightarrow g(W^\perp) = W^\perp$. So $U = W^\perp$ is a complementary invariant subspace. □

Alternative pf: choose any complementary subspace $U_0 \subset V$ st. $V = U_0 \oplus W$.
Let $\pi_0 : V \to W$ projection onto W with kernel U_0 ($\pi_{0|U_0} = 0$, $\pi_{0|W} = id$).
Define $\pi(v) = \frac{1}{|G|}\sum_{g\in G} g\pi_0(g^{-1}v) \in W$. Then $\pi : V \to W$ is a homomorphism of rep$^{\text{ns}}$ (ie. G-equivariant: $g\pi g^{-1} = \pi \ \forall g$), so $U = \ker \pi$ is an invariant subspace.
Since $\pi_{|W} = id$, π is surjective and $V = U \oplus W$ (dim/rank formula and $U \cap W = \{0\}$). □

Rmk: • the proof fails if $\mathrm{char}(k) \neq 0$ (more specifically $\mathrm{char}(k) = p \mid |G|$). this is one of the reasons that modular representations (= over fields of char >0) are more complicated.
• it also fails if G is infinite (and doesn't carry a finite invariant measure) as we can't use averaging trick. (Averaging works for compact Lie groups such as S^1, $SO(n)$, ...)

Ex: $G = \mathbb{Z}$ or $\mathbb{R}$ acting on $\mathbb{C}^2$ by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$
then the first factor $\mathbb{C}\times 0$ is invariant under G, but $\nexists$ complementary inv^t subspace.

Goal: given G, find its irreducible representations, describe how others decompose into irreducibles.

Schur's Lemma:
• If V, W are irreducible rep$^{\text{ns}}$ of G, and $\varphi: V \to W$ any homom. of representations, then either $\varphi = 0$, or φ is an isomorphism.
• Over $k = \mathbb{C}$: if V is irreducible and $\varphi : V \to V$ is a homom. of representations then φ is a multiple of identity.

Proof: • given $\varphi : V \to W$, $\mathrm{Ker}(\varphi)$ is an invariant subspace of V, ie. a subrepresentation. Since V is irreducible, either $\mathrm{Ker}(\varphi) = 0$ (φ injective) or $\mathrm{Ker}(\varphi) = V$ ($\varphi = 0$). Similarly, $\mathrm{Im}(\varphi) \subset W$ is invariant hence either zero ($\varphi = 0$) or W (φ surjective). Hence, either $\varphi = 0$ or φ is an isomorphism.
• over $k = \mathbb{C}$, any $\varphi : V \to V$ has an eigenvalue λ. Then $\varphi - \lambda I : V \to V$ is also equivariant, has nonzero kernel, hence $\varphi - \lambda I = 0$ by the above. Thus $\varphi = \lambda I$. □