Def: A *representation* of a group G is a vector space V + an action of G on V by linear operators: ie. $G \times V \to V$ st. $\forall g \in G$, $g : V \to V$ linear map.

Equivalently: a homomorphism $\rho : G \to GL(V)$ the group of invertible linear operators $V \to V$.

Def:
- A *subrepresentation* is a subspace $W \subset V$ which is invariant under G, ie. $gW = W \quad \forall g \in G$.
- A representation is *irreducible* if it has no nontrivial subrepresentations.

Ex: G finite abelian group $\Rightarrow$ every finite dim. rep of G over $\mathbb{C}$ is a direct sum of 1-dimensional sub-reps. Isom. classes of 1-dim. representations: $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$.

Def: Given two representations V, W of G, a *homomorphism of representations* $\varphi : V \to W$ is a linear map $\varphi : V \to W$ that is *equivariant*, ie. compatible with the group actions:
$$\varphi(gv) = g\varphi(v) \ \forall v \in V \ \forall g \in G.$$

Theorem: Let V be any rep. of a finite group G (over $\mathbb{C}$, or k of char. 0), and suppose $W \subset V$ is an invariant subspace (ie., subrepresentation). Then there exists another invariant subspace $U \subset V$ st. $V = U \oplus W$. (as a direct sum of rep's)

Corollary: any finite dim. representation of a finite gp decomposes into direct sum of irreducibles.

Two proofs of thm. The first one uses:

Lemma: If V is a $\mathbb{C}$-representation of a finite group G, then there exists a positive definite Hermitian inner product on V which is preserved by G: $H(gv, gw) = H(v, w) \quad \forall g, v, w$, ie. all the linear operators $g : V \to V$ are *unitary*.

Pf. Lemma: Let H_0 be any Hermitian inner product on V, and use averaging trick to set
$$H(v,w) = \frac{1}{|G|} \sum_{g \in G} H_0(gv, gw).$$
Then H is still Hermitian and definite positive (hence an inner product), and $H(gv, gw) = H(v, w)$. □

Pf. thm: Equip V with a G-invariant Hermitian inner product H as in the Lemma. Then if $g(W) = W$, g unitary $\Rightarrow g(W^\perp) = W^\perp$. So $U = W^\perp$ is a complementary invariant subspace. □

Alternative pf: choose any complementary subspace $U_0 \subset V$ st. $V = U_0 \oplus W$.
Let $\pi_0 : V \to W$ projection onto W with kernel U_0 ($\pi_{0|U_0} = 0$, $\pi_{0|W} = id$).
Define $\pi(v) = \frac{1}{|G|} \sum_{g \in G} g\pi_0(g^{-1}v) \in W$. Then $\pi : V \to W$ is a homomorphism of rep$^{\underline{ns}}$

(ie. G-equivariant ; $g\pi g^{-1}=\pi \ \forall g$), so $U=\ker\pi$ is an invariant subspace.
Since $\pi|_W = id$, π is surjective and $V=U\oplus W$ (dim/rank formula and $U\cap W=\{0\}$). □

Rmk: • the proof fails if $char(k)\neq 0$ (more specifically, $char(k)=p \mid |G|$). This is one of the reasons that modular representations (= over fields of char > 0) are more complicated.

• it also fails if G is infinite (and doesn't carry a finite invariant measure) as we can't use averaging trick. (Averaging works for compact Lie groups such as S^1, $SO(n)$, ...)

Ex: $G=\mathbb{Z}$ or $\mathbb{R}$ acting on $\mathbb{C}^2$ by $t\mapsto \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$
then the first factor $\mathbb{C}\times 0$ is invariant under G, but $\nexists$ complementary inv't subspace.

Goal: given G, find its irreducible representations, describe how others decompose into irreducibles.

Schur's Lemma:
• If V, W are irreducible rep^{ns} of G, and $\varphi: V\to W$ any homom. of representations, then either $\varphi=0$, or φ is an isomorphism.
• Over $k=\mathbb{C}$: if V is irreducible and $\varphi: V\to V$ is a homom. of representations then φ is a multiple of identity.

Proof: • given $\varphi: V\to W$, $Ker(\varphi)$ is an invariant subspace of V, ie. a subrepresentation. Since V is irreducible, either $Ker(\varphi)=0$ (φ injective) or $Ker(\varphi)=V$ ($\varphi=0$). Similarly, $Im(\varphi)\subset W$ is invariant hence either zero ($\varphi=0$) or W (φ surjective). Hence, either $\varphi=0$ or φ is an isomorphism.

• over $k=\mathbb{C}$, any $\varphi: V\to V$ has an eigenvalue λ. Then $\varphi-\lambda I: V\to V$ is also equivariant, has nonzero kernel, hence $\varphi-\lambda I=0$ by the above. Thus $\varphi=\lambda I$. □

Ex: Let V irred. rep of G, and $h\in Z(G)$ center of G (h commutes with $\forall g\in G$).
Then the action of $h: V\to V$ satisfies: $\forall g\in G$, $h(gv)=g\,h(v)$: so h is equivariant, ie. $h\in Hom_G(V,V)=\mathbb{C}.id$ by Schur's lemma $\Rightarrow$ h acts by a multiple of id.

In particular, if G is abelian and V is irreducible then every element of G acts by a multiple of id; this gives another proof that irred. rep^s of finite abelian groups are 1-dimensional.

Next we look at the simplest nonabelian group, S_3 ($=\mathfrak{S}_3$ in Fulton-Harris).
We know the trivial representation $U\simeq\mathbb{C}$ (every $\sigma\in S_3$ acts by id)
There's another 1-d. rep. $U'\simeq\mathbb{C}$ with the other element of $Hom(S_3,\mathbb{C}^*)$: the alternating rep^n (also called sign rep^n) where $\sigma\in S_3$ acts by $(-1)^\sigma$.

We also have the permutation representation $\simeq \mathbb{C}^3$ with basis e_1, e_2, e_3, on which S_3 acts by permutation matrices: σ maps $e_i \mapsto e_{\sigma(i)}$.

This has an invariant subspace, namely $\mathrm{span}(e_1+e_2+e_3)$, and we easily find a complementary sub-repⁿ, namely $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1+z_2+z_3 = 0\}$

This is called the <u>standard</u> representation of S_3, $\dim V = 2$, and it is irreducible.

<u>Rmk</u>: similarly for S_n: the two 1-dim. representations are the trivial rep. $U = \mathbb{C}$ and the alternating rep. $U' = \mathbb{C}$ with σ acting by $(-1)^\sigma$, and the permutation repⁿ $\mathbb{C}^n$ with σ acting by $e_i \mapsto e_{\sigma(i)}$ has an inv. subspace $\mathrm{span}(e_1 + \dots + e_n) \simeq U$, with complementary subrep. $V = \{(z_1 \dots z_n) \in \mathbb{C}^n \mid \Sigma z_i = 0\}$; it turns out V is irreducible - the standard rep. of S_n, with $\dim V = n-1$.

What is specific to S_3 is that this is the whole story (over $\mathbb{C}$). (S_n has more irred. repⁿˢ, in fact # irred. reps of S_n = $p(n)$ number of partitions of n ...).

<u>Prop</u>: U, U' and V are the only irreducible representations of S_3 (over $\mathbb{C}$). Hence, any rep of S_3 is isomorphic to a direct sum $U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$ for some (unique) $a, b, c \in \mathbb{N}$.

<u>Proof</u>: Let W be any (finite dim. / $\mathbb{C}$) representation of S_3. Restrict first to the abelian subgroup $A_3 \simeq \mathbb{Z}/3 \subset S_3$: let $\tau \in S_3$ be any 3-cycle, and $\sigma \in S_3$ any transposition. Then $\tau^3 = \sigma^2 = id$, and $\sigma^{-1}\tau\sigma = \tau^2$. Restricting the representation to the subgroup generated by τ ($\simeq \mathbb{Z}/3$), W has a basis of eigenvectors (v_j), where $\tau(v_j) = \lambda_j v_j$ where $\lambda_j = e^{2\pi i k_j/3}$ root of unity. Now let's see how σ acts.

If $v \in W$ is an eigenvector for τ, $\tau(v) = \lambda v$, then $\tau(\sigma v) = \sigma(\tau^2 v) = \lambda^2 \sigma(v)$.

So: σ maps the λ-eigenspace of τ to its λ^2-eigenspace.

(<u>Rmk</u>: $\forall v$ eigenvector of τ, $\mathrm{span}(v, \sigma v)$ is an invariant subspace, since both generators σ and τ preserve it. So now we know irred. reps. have $\dim \leq 2$)

Let's now specialize to the case W irreducible, and choose $v \in W$ an eigenvector of τ.

<u>Case $\lambda = 1$</u>: $\tau(v) = v$, and by the above, $\tau(\sigma(v)) = \sigma(v)$. If $\sigma(v)$ is linearly indep^t of v, then $w = v + \sigma(v) \neq 0$ satisfies $\sigma(w) = \sigma(v) + \sigma^2(v) = w$, and $\tau(w) = w$, so we get an invariant subspace (trivial subrepⁿ) $\mathrm{span}(w) \simeq U$. Contradicts irreducibility.

So $\sigma(v)$ is a scalar multiple of v; since $\sigma^2 = id$, $\sigma(v) = \pm v$.

In both cases, span(v) is invariant, and $\simeq U$ if $\sigma(v)=v$ $\tau(v)=v$
U' if $\sigma(v)=-v$ $\tau(v)=v$.

If W irreducible this is all of W.

Case $\lambda = e^{\pm 2\pi i/3}$: then $\tau(v)=\lambda v$ and $\tau(\sigma(v)) = \lambda^2 \sigma(v)$ by the above.

Since $\lambda \neq \lambda^2$, these two eigenvectors of τ are linearly independent;

span$(v, \sigma(v))$ is an invariant subspace, hence by irreducibility, equals W.

We check that $W \simeq V$ standard rep^n by mapping v to the λ-eigenvector of τ in the standard rep^n. (ie. $\{v, \sigma(v)\}$ map to $\{(1,\lambda^2,\lambda), (1,\lambda,\lambda^2)\} \subset V \subset \mathbb{C}^3$) $\square$

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* Given a representation of S_3, $W \simeq U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$, how do we find a,b,c?

A: Look at eigenvalues of τ: the 1-eigenspace of τ is $U^{\oplus a} \oplus U'^{\oplus b}$, so
$a+b = \dim \ker(\tau - 1)$; whereas the $e^{\pm 2\pi i/3}$-eigenspaces each have $\dim = c$.

So: multiplicities of eigenvalues of τ give $a+b$ and c.

Likewise, σ acts by $+1$ on U, -1 on U', and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on V,
so the eigenspaces of σ have dim. $a+c$ for 1, $b+c$ for -1.

From this we get a, b, and c.

Example: consider V the standard rep. of S_3, and $V^{\otimes 2} = V \otimes V$ also a rep^n
(recall: $g(v \otimes w) = gv \otimes gw$). How does $V^{\otimes 2}$ decompose into irreducibles?

Start with a basis e_1, e_2 of V with $\tau e_1 = \lambda e_1$, $\tau e_2 = \lambda^2 e_2$ where $\lambda = e^{2\pi i/3}$
$\sigma e_1 = e_2$, $\sigma e_2 = e_1$.

Then $V \otimes V$ has a basis $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$, $e_2 \otimes e_2$.

These are eigenvectors of τ, with eigenvalues λ^2, 1, 1, λ.

Moreover, on the 1-eigenspace span$(e_1 \otimes e_2, e_2 \otimes e_1)$, σ swaps these two, so
$e_1 \otimes e_2 \pm e_2 \otimes e_1$ is an eigenvector of σ with eigenvalue ± 1.

Hence $V \otimes V \simeq U \oplus U' \oplus V$.

Similarly $\mathrm{Sym}^2 V$: basis e_1^2, $e_1 e_2$, e_2^2 $\rightsquigarrow \mathrm{Sym}^2(V) \simeq U \oplus V$.
τ acts by λ^2, 1, λ

(whereas $\Lambda^2 V \simeq U'$, perhaps unsurprisingly considering det. vs sign).

Next time we'll discuss symmetric polynomials, then introduce characters as a tool to study representations.