Last time we saw how rep's of S_3 can be decomposed into irreducibles efficiently by looking at eigenspaces of the transformations by which certain elements of S_3 act.

Recall: the irred. representations of S_3 are
$$\begin{cases} \bullet \text{ trivial rep. } & U = \mathbb{C}, \quad \sigma \text{ acts by } 1 \\ \bullet \text{ alternating } & U' = \mathbb{C} \qquad\qquad\quad (-1)^\sigma \\ \bullet \text{ standard } & V = \{z_1+z_2+z_3=0\} \subset \mathbb{C}^3, \ \sigma \text{ permutes coords} \end{cases}$$

and in terms of action of $\tau = $ 3-cycle, $\sigma = $ transposition
$$\begin{cases} U: & \tau = id \quad \sigma = id \\ U' & \tau = id \quad \sigma = -id \\ V & \tau \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix}, \ \sigma \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \lambda = e^{2\pi i/3} \end{cases}$$

$\Rightarrow$ given any rep^n W of S_3, $W \simeq U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$,
the $+1$-eigenspace of $\tau : W \to W$ has dim. $a+b$, λ/λ^2-eigenspaces dim. c
the $+1$-eigenspace of σ has dim. $a+c$, -1-eigenspace dim. $b+c$.

Example: consider V the standard rep. of S_3, and $V^{\otimes 2} = V \otimes V$ also a rep^n
(recall: $g(v \otimes w) = gv \otimes gw$). How does $V^{\otimes 2}$ decompose into irreducibles?

Start with a basis e_1, e_2 of V with $\tau e_1 = \lambda e_1$, $\tau e_2 = \lambda^2 e_2$ where $\lambda = e^{2\pi i/3}$
$\sigma e_1 = e_2$, $\sigma e_2 = e_1$.

Then $V \otimes V$ has a basis $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$, $e_2 \otimes e_2$.
These are eigenvectors of τ, with eigenvalues λ^2, 1, 1, λ.

Moreover, on the 1-eigenspace $\mathrm{span}(e_1 \otimes e_2, e_2 \otimes e_1)$, σ swaps these two, so
$e_1 \otimes e_2 \pm e_2 \otimes e_1$ is an eigenvector of σ with eigenvalue ± 1.

Hence $V \otimes V \simeq U \oplus U' \oplus V$.

Similarly $\mathrm{Sym}^2 V$: basis e_1^2, $e_1 e_2$, e_2^2 $\rightsquigarrow \mathrm{Sym}^2(V) \simeq U \oplus V$.
τ acts by λ^2, 1, λ

(whereas $\Lambda^2 V \simeq U'$, perhaps unsurprisingly considering det. vs sign).

This generalizes to more complicated groups - we'll see that eigenvalues go a long way towards classifying representations - but we need some way of organizing the information.

Digression: Symmetric polynomials: (this is all motivation for the study of characters).

- Observe: an efficient way to store information about n (complex) numbers, unordered and possibly with repetitions, is to specify the coefficients of the polynomial of which they are the roots, ie. $\prod_{i=1}^n (x - \lambda_i)$. These coefficients are symmetric polynomials in $\lambda_1 \dots \lambda_n$

- S_n acts on the space of polynomials $\mathbb{C}[z_1, \dots, z_n]$ by permuting the variables.

Def: A <u>symmetric polynomial</u> is $f \in \mathbb{C}[z_1, \dots, z_n]$ that is a fixed point of the S_n-action, $\sigma(f) = f \quad \forall \sigma \in S_n$.

(<u>Rmk</u>: equality of polynomials means, as usual, equality of coefficients, which over a finite field is a stronger condition than having equality as functions on k^n. Of course over $\mathbb{C}$ no difference.)

Def: The <u>elementary symmetric polynomials</u>: $\sigma_1(z_1, \dots, z_n) = \sum_{i=1}^{n} z_i$,

$$\sigma_2(z_1, \dots, z_n) = \sum_{1 \le i < j \le n} z_i z_j, \quad \dots, \quad \sigma_k = \sum_{1 \le i_1 \le \dots \le i_k \le n} z_{i_1} \cdots z_{i_k}, \quad \dots \quad \sigma_n = \prod_{i=1}^{n} z_i.$$

<u>Check</u>: the coefficient of x^{n-k} in $\prod_{i=1}^{n}(x - z_i)$ is, up to sign $(-1)^k$, $\sigma_k(z_1, \dots, z_n)$.

<u>Hence</u>: the fundamental theorem of algebra gives a bijection

$$\{\text{unordered } n\text{-tuples of complex numbers, repetitions allowed}\} \overset{\sim}{\longleftrightarrow} \underset{\text{ordered tuples}}{\mathbb{C}^n}$$

$$[z_1, \dots, z_n] \longmapsto (\sigma_1(z_i), \dots, \sigma_n(z_i))$$

$$[\text{the roots of } x^n - \sigma_1 x^{n-1} + \dots + (-1)^n \sigma_n] \longleftarrow (\sigma_1, \dots, \sigma_n)$$

In other terms: $[z_1, \dots, z_n] \longleftrightarrow$ coefficients of the polynomial $\prod(x - z_i)$.

<u>Theorem</u>: The subring of symmetric polynomials in $\mathbb{C}[z_1 \dots z_n]$, ie. $\mathbb{C}[z_1 \dots z_n]^{S_n}$, is isomorphic to the polynomial algebra in n variables $\mathbb{C}[\sigma_1, \dots, \sigma_n]$.
Ie. every symmetric polynomial is uniquely a polynomial expression in the elementary symmetric polynomials.

* We won't prove this, but to see why this works, look at the case $n = 2$.
The vector space of symmetric polynomials has basis

$$1 = 1$$
$$z_1 + z_2 = \sigma_1$$
$$z_1^2 + z_2^2 = (z_1 + z_2)^2 - 2z_1 z_2 = \sigma_1^2 - 2\sigma_2$$
$$z_1 z_2 = \sigma_2$$

$$z_1^3 + z_2^3 = \sigma_1^3 - 3z_1^2 z_2 - 3z_1 z_2^2 = \sigma_1^3 - 3\sigma_1\sigma_2$$
$$z_1^2 z_2 + z_1 z_2^2 = \sigma_1 \sigma_2$$
$$\dots\dots$$

Observe: any symmetric polynomial in 2 variables can be written as

$$p(z_1, z_2) = \sum a_k (z_1^k + z_2^k) + z_1 z_2 \, q(z_1, z_2)$$
$$= \sum a_k (z_1 + z_2)^k + z_1 z_2 \, q'(z_1, z_2)$$
$$= \sum a_k \sigma_1^k + \sigma_2 \cdot q'$$

& work by induction on degree.

Rmk: the theorem can be understood in terms of rep theory of S_n!
Namely, the space of homogeneous deg. 1 polynomials is $W_1 = \mathrm{span}(z_1, \dots, z_n) \simeq \mathbb{C}^n$ on which S_n acts by permutation rep. $\cong V \oplus U$ (standard $\oplus$ trivial) and the invariant part is $W_1^{S_n} \simeq U$ trivial summand. Now, homogeneous deg. d polynomials are $W_d = \mathrm{Sym}^d(W_1)$, and the invariant part $W_d^{S_n}$ = trivial summands in the decomp. of W_d into irreducibles! (Unfortunately we haven't studied rep^{ns} of S_n in enough depth to carry through with a proof along these lines).

* Another family of symmetric polynomials are the power sums:

$$\tau_k(z_1, \dots, z_n) = \sum_{i=1}^n z_i^k. \qquad \tau_1 = \sigma_1, \quad \tau_2 = \sigma_1^2 - 2\sigma_2, \dots$$

These make sense for all k, but in fact $\tau_1, \dots, \tau_n$ suffice:

Thm': $\mathbb{C}[z_1, \dots, z_n]^{S_n} \cong \mathbb{C}[\tau_1, \dots, \tau_n]$

In particular: specifying an unordered tuple $\{z_1 \dots z_n\}$ is equivalent to specifying $\sum z_i$, $\sum z_i^2, \dots, \sum z_i^n$.

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* Back to representation theory – why we care about this:
We've seen that, to understand a representation V of G, we should look at the eigenvalues of $g: V \to V$ for each $g \in G$; but this is a lot of information.
We've just said: to specify the eigenvalues λ_i of $g: V \to V$, it is enough to specify the power sums $\sum \lambda_i^k$. But in fact $\sum \lambda_i^k = \mathrm{tr}(g^k)$!
So it's enough to describe just the sum of the eigenvalues $\sum \lambda_i = \mathrm{tr}(g)$ for every $g \in G$ – since G is a group, the trace of g^k is also part of this.

Def. The character χ_V of a representation V is the function $\chi_V : G \to \mathbb{C}$, $\chi_V(g) = \mathrm{tr}(g)$.

Remark: for a 1-dim^l representation of G, ie. a homom. $G \to \mathbb{C}^*$, the character is just the same thing, hence a (multiplicative) homom.
For a higher-dim^l representation, though, $\chi(g_1 g_2) \neq \chi(g_1)\chi(g_2)$.

However, since trace is conjugation invariant, $\mathrm{tr}(ghg^{-1}) = \mathrm{tr}(h)$,
so $\chi_V(g)$ only depends on the conjugacy class of g.

Def. A class function $f: G \to \mathbb{C}$ is a function invariant under conjugation, $f(ghg^{-1}) = f(h)$.

Ex: given representations V and W:

- $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$ (eigenvalues of $\left(\begin{array}{c|c}\varphi & 0\\ \hline 0 & \psi\end{array}\right)\ldots$)
- $\chi_{V\otimes W}(g) = \chi_V(g)\,\chi_W(g)$ (eigenvalues of $\varphi\otimes\psi : v_i\otimes w_j \mapsto \lambda_i\lambda_j'\, v_i\otimes w_j$)
- $\chi_{V^*}(g) = \overline{\chi_V(g)}$ since g acts by ${}^t(g^{-1})$, and eigenvalues are roots of unity so $\lambda_i^{-1} = \overline{\lambda_i} \Rightarrow \Sigma\lambda_i^{-1} = \Sigma\overline{\lambda_i}$
- $\chi_{\Lambda^2 V}(g) = \sum_{i<j}\lambda_i\lambda_j = \frac{1}{2}\left((\Sigma\lambda_i)^2 - \Sigma\lambda_i^2\right) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right)$

Ex: If G acts on a finite set S, then there is an associated permutation representation V of dimension $|S|$, with basis $(e_s)_{s\in S}$, G acts by permutation matrices $g\cdot e_s = e_{g\cdot s}$.
Then $\chi_V(g) = tr(g) = \#\{s\in S \mid g\cdot s = s\}$, since 1's on diagonal of matrix correspond to fixed points of g, and 0's otherwise.

The <u>character table</u> of a group = list, for each irred. rep^n of G, the values of ~~the~~ its character on each conjugacy class of G.

<u>Example:</u> $G = S_3$:

	e	(12)	(123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

$\longrightarrow$ conjugacy classes

$\downarrow$ irred. rep^{ns}

$\chi_V(e) = tr(id) = \dim V$.

$\leftarrow$ <u>either</u> from eigenvalues ± 1 for (12), $e^{\pm 2\pi i/3}$ for (123)
<u>or</u> $U\oplus V$ = permutation representation takes values #fixed pts $= (3,1,0)$ then subtract $\chi_U = (1,1,1)$.

Now we have a faster way of decomposing $V\otimes V$ into irreducibles:

$\chi_{V\otimes V}(g) = \chi_V(g)^2$ so $\chi_{V\otimes V}$ takes values $(4,0,1)$

$\chi_U, \chi_{U'}, \chi_V$ are linearly independent, $\chi_{V\otimes V} = \chi_U + \chi_{U'} + \chi_V \Rightarrow V\otimes V \simeq U\oplus U'\oplus V$.

(This is equivalent to counting eigenvalues as we did last time, but somewhat faster!)

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* Now for some magic with characters...

- If V is a representation of G, the invariant part is $V^G = \{v\in V \mid gv = v\ \forall g\in G\}$.

<u>Prop:</u> $\varphi = \frac{1}{|G|}\sum_{g\in G} g : V\to V$ is a projection onto $V^G\subset V$: $\begin{cases} Im(\varphi) = V^G \\ \varphi_{|V^G} = id.\end{cases}$

- <u>So:</u> $\dim(V^G) = tr(\varphi) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$.
- If V, W are rep^s of G, $Hom_G(V,W) = Hom(V,W)^G = (V^*\otimes W)^G$, so:

G-action on $Hom(V,W)$ is $g(\varphi) = g\varphi g^{-1}$!

$\dim Hom_G(V,W) = \frac{1}{|G|}\sum_{g\in G}\chi_{V^*\otimes W}(g) = \frac{1}{|G|}\sum_g \overline{\chi_V(g)}\,\chi_W(g)$ ... more next time.