Recall: Def: The character χ_V of a representation V is the function $\chi_V : G \to \mathbb{C}$, $\chi_V(g) = tr(g)$.

χ_V is a class function on G, ie. $\chi_V(g)$ only depends on the conjugacy class of g.

Ex: given representations V and W:

- $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$ (eigenvalues of $\left(\begin{array}{c|c} \varphi & 0 \\ \hline 0 & \psi \end{array}\right) \dots$)
- $\chi_{V\otimes W}(g) = \chi_V(g)\,\chi_W(g)$ (eigenvalues of $\varphi\otimes\psi : v_i\otimes w_j \mapsto \lambda_i\lambda_j' \, v_i\otimes w_j$)
- $\chi_{V^*}(g) = \overline{\chi_V(g)}$ since g acts by ${}^t(g^{-1})$, and eigenvalues are roots of unity so $\lambda_i^{-1} = \bar\lambda_i \Rightarrow \sum \lambda_i^{-1} = \sum \bar\lambda_i$
- hence $\chi_{Hom(V,W)}(g) = \overline{\chi_V(g)}\,\chi_W(g)$.

The character table of a group = list, for each irred. rep$^{\underline{n}}$ of G, the values of the its character on each conjugacy class of G.

Example: $G = S_3$:

irred. rep$^{\underline{ns}}$ ↓	e	(12)	(123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

→ conjugacy classes

$\chi_V(e) = tr(id) = \dim(V)$

-1 ← either from eigenvalues ± 1 for (12), $e^{\pm 2\pi i/3}$ for (123)

or $U\oplus V$ = permut. rep$^{\underline{n}}$, has χ = #fixed points = $(3,1,0)$ then subtract $\chi_U = (1,1,1)$.

Last time we decomposed $V\otimes V$ into irreducibles "by hand", now we can do faster:

$\chi_{V\otimes V}(g) = \chi_V(g)^2$ so $\chi_{V\otimes V}$ takes values $(4,0,1)$

$\chi_U, \chi_{U'}, \chi_V$ are linearly independent, $\chi_{V\otimes V} = \chi_U + \chi_{U'} + \chi_V \Rightarrow V\otimes V \simeq U\oplus U'\oplus V$.

- If V is a representation of G, the invariant part is $V^G = \{v\in V \mid gv = v \ \forall g\in G\}$.

Prop: $\varphi = \frac{1}{|G|}\sum_{g\in G} g : V \to V$ is a projection onto $V^G \subset V$: $\begin{cases} Im(\varphi) = V^G \\ \varphi_{|V^G} = id. \end{cases}$

- So: $\dim(V^G) = tr(\varphi) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$.
- If V, W are rep^{s} of G, $Hom_G(V,W) = Hom(V,W)^G = (V^*\otimes W)^G$, so:

  G-action on $Hom(V,W)$ is $g(\varphi) = g\varphi g^{-1}$.

  $\dim Hom_G(V,W) = \frac{1}{|G|}\sum_{g\in G}\chi_{V^*\otimes W}(g) = \frac{1}{|G|}\sum_g \overline{\chi_V(g)}\,\chi_W(g) \quad \dots$

but if V and W are irreducible, then by Schur's lemma, $\dim Hom_G(V,W) = \begin{cases} 1 & \text{if } V\simeq W \\ 0 & \text{else.} \end{cases}$

Def. Define a Hermitian inner product on the space of class functions $G \to \mathbb{C}$ by
$$H(\alpha,\beta) = \frac{1}{|G|}\sum_{g\in G} \bar{\alpha}(g)\beta(g).$$

For characters of rep^{ns}, by the above, $\dim \mathrm{Hom}_G(V,W) = H(\chi_V, \chi_W)$.

$\Rightarrow$ **Thm:** The characters of irreducible representations of G are orthonormal for H.

This implies characters of irred. rep^s are linearly independent class functions!

Corollary: 1. The number of irreducible representations of G is at most the number of conjugacy classes of G. (We'll see later that they are in fact equal).

Corollary: 2. Every representation of G is completely determined by its character: denoting the irred. reps by $V_1,\dots,V_k$, any rep^n $W \simeq \bigoplus V_i^{\oplus a_i}$, where
$$a_i = \dim \mathrm{Hom}_G(V_i, W) = H(\chi_{V_i}, \chi_W).$$

Corollary: 3. For any rep^n $W = \bigoplus V_i^{\oplus a_i}$, $H(\chi_W,\chi_W) = \sum a_i^2$, and W is irreducible iff $H(\chi_W,\chi_W) = 1$.

This is useful because, given a rep^n W, it gives info about # irreducible summands making up V. Eg:

$H(\chi_W,\chi_W) =$		
1	$\Leftrightarrow$ $W =$	irreducible
2		direct sum of 2 different irred's.
3		—"— 3 —"—
4		either 4 different, or twice the same.

* We now apply this to the <u>regular representation</u> R = vector space with basis $\{e_g\}_{g\in G}$ and G acts by permuting basis vectors by left multiplication: $g\cdot e_h = e_{gh}$.

Now let $V_1,\dots,V_k$ be the irreducible rep^s of G, and write $R = \bigoplus V_i^{\oplus a_i}$. What are the a_i?

Since G acts by permutation matrices, $\chi_R(g) = tr(g) = \#\{h\in G \,/\, g\cdot e_h = e_h\}$ but unless $g=e$ there are no fixed points $\Rightarrow \chi_R(g) = \begin{cases} |G| & \text{if } g=e \\ 0 & g\neq e.\end{cases}$

So $H(\chi_R, \chi_{V_i}) = \frac{1}{|G|}\sum_g \overline{\chi_R(g)}\,\chi_{V_i}(g) = \chi_{V_i}(e) = tr(id_{V_i}) = \dim V_i$.

Hence each V_i appears $a_i = \dim V_i$ times in the regular representation R.

And now Cor. 3 $\Rightarrow H(\chi_R,\chi_R) = |G| = \sum a_i^2 = \sum (\dim V_i)^2$.

direct calc: $\frac{1}{|G|}\sum_g |\chi_R(g)|^2 = \frac{1}{|G|}|\chi_R(e)|^2 = |G|$

<u>Corollary 4:</u> The irreducible representations $V_1, \dots, V_k$ of G satisfy $\sum (\dim V_i)^2 = |G|$.

At this point we actually have a lot of info about the irred. rep.^{ns} of G & their characters.

<u>Example:</u> $G = S_4$. The conjugacy classes: $\{e\}$ size 1, transpositions size 6, 3-cycles (8), 4-cycles (6), pairs of transpositions (3).

We know 3 irred. reps: U = trivial, U' = alternating, V = standard.

Character table:

	1	6	8	6	3	
	e	(12)	(123)	(1234)	$(12)(34)$	
U	1	1	1	1	1	$\leftarrow$ g acts by id, $\mathrm{tr}=1$.
U'	1	-1	1	-1	1	$\leftarrow$ $\mathrm{tr}\,(-1)^\sigma = (-1)^\sigma$.
V	3	1	0	-1	-1	

to find this one: $U \oplus V$ = permutation representation $\mathbb{C}^4$,

$\chi_{U\oplus V}(\sigma) = \mathrm{tr}(\sigma) = \#\text{fixed points} = \#\{i \,/\, \sigma(i) = i\}$ $\Rightarrow$ $\chi_V(\sigma) = \#\text{fix pts} - 1$.

<u>Quick check:</u> these are indeed orthonormal!

However: $\sum \dim^2 = 1^2 + 1^2 + 3^2 = 11 < 24$ $\Rightarrow$ there are other irred. rep.^{ns}!

in fact:
- corollary <u>1</u> says we're missing at most two (# irred. reps. $\leq$ # conjugacy classes $= 5$)
- since we're missing 13 which is not a square, we're missing exactly two, of dim's. 2 and 3 ($\Rightarrow \sum \dim^2 = 24$)

* How do we build the missing entries? Start by looking at tensor products of known reps.

For a start, the tensor product of an irred. rep. with a 1-dimensional rep. is still irreducible ($\otimes$ 1-dim. rep. has "same" invariant subspaces), so we can look at

$V' = V \otimes U'$ (twist standard rep. by $(-1)^\sigma$), has $\chi_{V'} = \chi_V \cdot \chi_{U'} = (3, -1, 0, 1, -1)$,

this is indeed irreducible ($H(\chi_{V'}, \chi_{V'}) = 1$) and different from V!

We have one <u>last</u> 2dim^l irred. rep. W to find!

Since $W \otimes U'$ is also a 2d irred. rep., necessarily $W \otimes U' \simeq W$. This implies

$\chi_W = \chi_W \chi_{U'}$ ie. $\chi_W = 0$ on the odd conjugacy classes ((12) and (1234))

The orthogonality relations allow us to find the rest of χ_W without having constructed it!

	1	6	8	6	3
	e	(12)	(123)	(1234)	$(12)(34)$
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	$a=-1$	0	$b=2$

$H(\chi_U,\chi_W)=\frac{1}{24}(2+8a+3b)=0$, $H(\chi_V,\chi_W)=\frac{1}{24}(6-3b)=0 \Rightarrow b=2,\ a=-1.$

Note that $\chi_W((12)(34))=2$ means the eigenvalues are 1 and 1! (roots of unity, summing to 2)

This gives a big clue about W: the normal subgroup $H=\{id\}\cup\{(ij)(kl)\}\simeq \mathbb{Z}/2\times\mathbb{Z}/2$ is in the kernel of $S_4 \xrightarrow{\rho} GL(W)$, ie. ρ factors through the quotient $S_4/H\simeq S_3$.
(recall: S_4 acts on the set of splittings of $\{1,2,3,4\}$ into 2 pairs - there are 3 of those).
Under this quotient, transpositions, 4-cycles $\longmapsto$ transpositions, 3-cycles $\mapsto$ 3-cycles,
and the character χ_W becomes $\left\{\begin{array}{l} id\mapsto 2 \\ \text{transp}\mapsto 0 \\ \text{3-cycle}\mapsto -1\end{array}\right\}$ - this is the standard rep. of S_3! "pulled back" to S_4 by $S_4\twoheadrightarrow S_3$.

* The other option to construct W is to look at $V\otimes V$: $\chi_{V\otimes V}=\chi_V^2=(9,1,0,1,1)$
We have $H(\chi_U,\chi_{V\otimes V})=1$, $H(\chi_{U'},\chi_{V\otimes V})=0$, $H(\chi_V,\chi_{V\otimes V})=\frac{1}{24}(27+6-6-3)=1$,
$H(\chi_{V'},\chi_{V\otimes V})=\frac{1}{24}(27-6+6-3)=1$, so $V\otimes V$ contains $U\oplus V\oplus V'$ (dim. 7)
and this leaves us one copy of the missing irreducible W. So: $V\otimes V=U\oplus V\oplus V'\oplus W$
(and we can find χ_W by subtracting the others from $\chi_{V\otimes V}$).

<u>Ex</u>: A_4 alternating subgroup of S_4. This has 4 conjugacy classes: $\{e\}$ 1 element
(3-cycles are one conj class in S_4 but split in A_4, see lecture 23) $\left\{\begin{array}{ll}(123) & 4\\ (132) & 4\end{array}\right.$
$(12)(34)$ 3

→ We can start by restricting to A_4 the irred. rep's of S_4 - some become isomorphic (eg the alternating rep. U' has elements of A_4 acting by $(-1)^6=1$ so $\simeq$ trivial), others might become reducible. This is feasible but tricky (largely W's fault).

→ Or we can go at it directly! We know there's at most 4 irred. reps, of $\sum \dim^2 = 12$, including the trivial rep^n of dim 1 $\Rightarrow$ the only option is $12=3^2+1^2+1^2+1^2$.

The three 1-dim^{l} representations correspond to $\mathrm{Hom}(A_4, \mathbb{C}^*) \ni id$ (trivial rep) and two other elements...

Observe $H = \{id\} \cup \{(ij)(kl)\}$ normal subgroup,

$A_4/H \simeq \mathbb{Z}/3$, so this gives the answer: $\mathrm{Hom}(A_4, \mathbb{C}^*) \simeq \widehat{\mathbb{Z}/3}$

$= \{m \mapsto e^{2\pi i m k/3}\}$.

Concretely, let $\lambda = e^{2\pi i/3}$, then the rank 1 rep's are:

	e	(123)	(132)	$(12)(34)$
U	1	1	1	1
U'	1	λ	λ^2	1
U''	1	λ^2	λ	1
V	3	0	0	-1

$\}$ $(ij)(kl) \in H$ act by id

(Note: $W_{|A_4} \simeq U' \oplus U''$.)

and the last one by orthogonality is: V. This is the restr. to A_4 of the standard rep. of S_4!