Today we'll look at rep's of S_5 and A_5, for extra practice with characters + to motivate discussion of restriction & induction of representations between G & subgroups.

One can start building the character table of S_5 the usual way: start with known rep's.

First we have U (trivial) and U' (alternating), and V (standard rep., dim 4).

Use: $V\oplus U \simeq$ permutation rep. $\mathbb{C}^5$, so $\chi_{V\oplus U}(\sigma) = \#\{i \,/\, \sigma(i)=i\}$, $\chi_V = \chi_{U\oplus V} - 1$.

	1	10	20	30	24	15	20
	e	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(123)(45)$
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
$V'=V\otimes U'$	4	-2	1	0	-1	0	1

Then we need more. Since $|S_5| = 120 = \sum \dim^2$, we're still missing 3 irreducibles with $\sum \dim^2 = 86$; the most effective way to find them is to keep building tensor products – namely look at $V\otimes V$ (dim. 16), or rather its two pieces $\mathrm{Sym}^2 V$ (dim. 10) and $\Lambda^2 V$ (dim. 6).

$\star$ Observe: if $g: V\to V$ has eigenvalues λ_i ($gv_i = \lambda_i v_i$, $1\le i\le r$) then the corresponding map on $\mathrm{Sym}^2 V$ has eigenvalues $\lambda_i\lambda_j$, $1\le i\le j\le r$ (recall: (v_i) basis of $V \Rightarrow (v_i v_j)$ basis of $\mathrm{Sym}^2 V$)
$\Lambda^2 V$ — $\lambda_i\lambda_j$, $1\le i<j\le r$ — $(v_i\wedge v_j)$ — $\Lambda^2 V$

Now,
$$\left.\begin{aligned}\sum_{i<j} \lambda_i\lambda_j &= \tfrac12\Big(\big(\textstyle\sum\lambda_i\big)^2 - \sum\lambda_i^2\Big)\\ \sum_{i\le j}\lambda_i\lambda_j &= \tfrac12\Big(\big(\textstyle\sum\lambda_i\big)^2 + \sum\lambda_i^2\Big)\end{aligned}\right\} \text{ so } \begin{aligned}\chi_{\Lambda^2 V}(g) &= \tfrac12\big(\chi_V(g)^2 - \chi_V(g^2)\big)\\ \chi_{\mathrm{Sym}^2 V}(g) &= \tfrac12\big(\chi_V(g)^2 + \chi_V(g^2)\big).\end{aligned}$$

(this is true for any rep$^{\text{n}}$).

This formula lets us calculate $\chi_{\Lambda^2 V}$ and $\chi_{\mathrm{Sym}^2 V}$ for the standard rep. of S_5.

	1	10	20	30	24	15	20
	e	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(123)(45)$
V	4	2	1	0	-1	0	-1
$\Lambda^2 V$	6	0	0	0	1	-2	0
$\mathrm{Sym}^2 V$	10	4	1	0	0	2	1

Observe: $H(\chi_{\Lambda^2 V}, \chi_{\Lambda^2 V}) = \frac{1}{120}\left(6^2 + 24 + 15\cdot 2^2\right) = 1$, so $\Lambda^2 V$ is irreducible!

whereas $H(\chi_{\mathrm{Sym}^2 V}, \chi_{\mathrm{Sym}^2 V}) = \frac{1}{120}\big(\underbrace{10^2 + 10\cdot 4^2 + 20 + 15\cdot 2^2 + 20}_{360}\big) = 3$

so $Sym^2 V$ splits into 3 irreducible summands.

$$H(\chi_U, \chi_{Sym^2 V}) = \frac{1}{120}\left(10 + 10.4 + 20 + 15.2 + 20\right) = 1 \Rightarrow \text{ one copy of } U$$

similar calculations $\Rightarrow$ $Sym^2 V$ also contains V with mult. 1; not U' or V'.

Hence $Sym^2 V = U \oplus V \oplus W$ for some irred. 5-dim! representation W.

Subtracting, we find χ_W - and one more, $W' = W \otimes U'$, which completes the list.

	1 e	10 (12)	20 (123)	30 (1234)	24 (12345)	15 $(12)(34)$	20 $(123)(45)$
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
$V' = V \otimes U'$	4	-2	1	0	-1	0	1
$\Lambda^2 V$	6	0	0	0	1	-2	0
$(U \oplus V \oplus W = Sym^2 V$	10	4	1	0	0	2	$1)$
W	5	1	-1	-1	0	1	1
$W' = W \otimes U'$	5	-1	-1	1	0	1	-1

<u>Remark</u>: the standard rep! V and its exterior powers $\Lambda^2 V$, $\Lambda^3 V \simeq V'$, and $\Lambda^4 V \simeq U'$ are all irreducible! This is in fact a general property - $\forall\, 0 \le k \le n-1$, the exterior powers $\Lambda^k V$ of the standard rep. of S_n are all irreducible (see Fulton-Harris §3.2).

- Next, move on to A_5. Starting point = restrict irreducible representations of S_5 to A_5 and see which ones remain irreducible or decompose. Of course different irred. reps. of S_5 can become isomorphic after restriction - namely elements of A_5 act by id on U' so U' becomes trivial! and the restrictions of V and $V' = V \otimes U'$ become isomorphic, similarly W. The character table for S_5 gives, after restriction:

	1 e	20 (123)	12 (12345)	12 (12354)	15 $(12)(34)$
U	1	1	1	1	1
V	4	1	-1	-1	0
W	5	-1	0	0	1
$\Lambda^2 V$	6	0	1	1	-2

Calculating $H(\chi, \chi)$ we find that U, V, W are irreducible, while $H(\chi_{\Lambda^2 V}, \chi_{\Lambda^2 V}) = 2$ so $\Lambda^2 V$ breaks into the direct sum of 2 distinct irreducibles. Also $\Lambda^2 V$ doesn't contain U, V or W, so $\Lambda^2 V = Y \oplus Z$ the last two irreducible rep's of A_5.

From $\sum \dim^2 = |A_5| = 60$ we find $\dim Y = \dim Z = 3$. How do we find χ_Y and χ_Z?

Using orthogonality and $\chi_Y + \chi_Z = \chi_{\Lambda^2 V}$, so $\chi_Y - \chi_Z \in \mathrm{span}(\chi_U, \chi_V, \chi_W, \chi_{\Lambda^2 V})^{\perp}$

Hence $\chi_Y - \chi_Z = (0,0,a,-a,0)$, where $H(\chi_Y - \chi_Z, \chi_Y - \chi_Z) = 2 \Rightarrow 24a^2 = 120$, $a = \pm\sqrt{5}$.

Thus:

	1 e	20 (123)	12 (12345)	12 (12354)	15 $(12)(34)$
U	1	1	1	1	1
V	4	1	-1	-1	0
W	5	-1	0	0	1
Y	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
Z	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1

What are Y and Z?? Recall: A_5 = rotational symmetries of an icosahedron in $\mathbb{R}^3$.

So: $A_5 \hookrightarrow SO(3) \subset GL(3,\mathbb{R}) \subset GL(3,\mathbb{C})$. ($Y$ and Z differ by an outer automorphism of A_5: conjugation by transposition inside S_5)

(The fact that the character takes irrational values implies that there does not exist a regular icosahedron (or dodecahedron) in $\mathbb{R}^3$ whose vertices all have rational coordinates! Otherwise we'd get that the representation factors through $GL(3,\mathbb{Q})$, and $\mathrm{tr}(g) \in \mathbb{Q}\ \forall g$)

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* <u>More systematic approach</u>: if G is a finite group and $H \subset G$ a subgroup, then we have a <u>restriction</u> operation Res_H^G: rep^{ns} of $G \longrightarrow$ repr^{ns} of H

This is actually a <u>functor</u> $\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(H)$ [objects = rep^s of G, of H; mor = homomorphisms of rep^s]

How about the opposite direction?

Suppose V is a rep. of G, and $W \subset V$ is invariant under H (but not all of G).

Now for $g \in G$, the subspace $gW \subset V$ depends only on the coset gH, and each gW is a rep^n of gHg^{-1}, with

$$\begin{array}{ccc} H & \xrightarrow{\rho} & GL(W) \\ c_g \downarrow \simeq & & \downarrow \text{conj. by } g \\ gHg^{-1} & \longrightarrow & GL(gW) \end{array}$$

The simplest possible scenario: $V = \bigoplus_{\sigma \in G/H} \sigma W$.

($\Rightarrow \dim V = |G/H| \cdot \dim W$) [in general there is no reason for this to hold].

If this happens, then the rep. of G is completely determined by that of H.

Indeed, choose representatives $\sigma_1, \dots, \sigma_k \in G$ of the cosets of H (each coset $\ni$ one σ_i)

given $g \in G$, $g\sigma_i \in \sigma_j H$ for some j, so there exists $h \in H$ st. $g = \sigma_j h \sigma_i^{-1}$.

Then g acts by mapping $\sigma_i W$ to $\sigma_j W$, with $g(\sigma_i w) = \sigma_j h(w)$.

Def: A representation V of G, with a subspace $W \subset V$ which is invariant under the subgroup $H \subset G$ (ie. a subrep. of $\mathrm{Res}_H^G V$), is said to be *induced* by $W \in \mathrm{Rep}(H)$ if, as a vector space, $V = \bigoplus_{\sigma \in G/H} \sigma W$. Write $V = \mathrm{Ind}_H^G W$.

ie. fixing one element in each coset, $\sigma_1, \dots, \sigma_k \in G$, we can write each $v \in V$ uniquely as $v = \sigma_1 w_1 + \dots + \sigma_k w_k$ for $w_1, \dots, w_k \in W$.

Thm: Given a representation W of H, the induced representation $V = \mathrm{Ind}_H^G W$ exists and is unique up to isomorphism of G-rep^s

Pf:
- Uniqueness: given $V \in \mathrm{Rep}(G)$ and $W \subset V$ invariant under H & s.t. $V = \bigoplus_{i=1}^k \sigma_i W$, necessarily $g \in G$ acts by mapping $\sigma_i W$ to $\sigma_j W$, where j is such that $g\sigma_i \in \sigma_j H$, ie. $h = \sigma_j^{-1} g \sigma_i \in H$, and necessarily $g(\sigma_i w) = \sigma_j h w \in \sigma_j W$. This determines the G-action uniquely.
- Existence: build $V = \bigoplus_{i=1}^k \sigma_i W$ where the σ_i are now formal symbols (ie. the direct sum of $k = |G/H|$ copies of W), and make $g \in G$ act as above. $\square$.

<u>Examples:</u> 1) The permutation rep. associated to the left action of G on G/H is induced by the trivial representation of H. Indeed V has a basis $\{e_\sigma\}_{\sigma \in G/H}$; the basis element e_H (for the coset H) is fixed by H, so $W = \mathrm{span}(e_H)$ is invariant under H, and $gW = \mathrm{span}(e_{gH})$, with

$$V = \bigoplus_{gH \in G/H} \mathrm{span}(e_{gH}) = \bigoplus_{gH \in G/H} gW.$$

2) The regular rep. of G is induced by the regular rep. of H: here $W = \mathrm{span}\{e_h, h \in H\} \subset V = \mathrm{span}\{e_g, g \in G\}$.

- <u>Fact</u>: $\mathrm{Ind}_H^G(W \oplus W') = \mathrm{Ind}_H^G(W) \oplus \mathrm{Ind}_H^G(W')$, but $\mathrm{Ind}(W \otimes W') \neq \mathrm{Ind}(W) \otimes \mathrm{Ind}(W')$.

On the other hand, if U is a rep. of G and W a rep. of H, then

$$\mathrm{Ind}(\mathrm{Res}(U) \otimes W) = U \otimes \mathrm{Ind}(W).$$

(indeed: $\mathrm{Ind}(W) = \bigoplus_{\sigma \in G/H} \sigma W$, so $U \otimes \mathrm{Ind}(W) = \bigoplus_{\sigma \in G/H} (U \otimes \sigma W) = \bigoplus_{G/H} \sigma(U \otimes W)$, where $U \otimes W \subset U \otimes \mathrm{Ind}(W)$ is invariant under H and $= \mathrm{Res}(U) \otimes W$ as H-rep^{n}).

<u>in particular</u>: $\mathrm{Ind}(\mathrm{Res}(U)) = U \otimes \mathrm{Ind}(\text{trivial}) = U \otimes (\text{permut. rep. } G/H)$.