Lec. 1 Artin ch. 2
- A group $(G,\cdot)$ is a set with an operation $\cdot : G\times G\to G$ st. (1) $\exists e\in G$ identity s.t. $eg=ge=g\ \forall g\in G$, (2) $\forall g\in G\ \exists$ inverse $g^{-1}\in G$ st. $gg^{-1}=g^{-1}g=e$, (3) associativity $(ab)c=a(bc)\ \forall a,b,c\in G$.
- A group is abelian if $\cdot$ is commutative ($ab=ba\ \forall a,b\in G$)
- Ex: $(\mathbb{Z},+)$, $(\mathbb{Z}/n,+)$, $(\mathbb{C}^*,\times)$, symmetric group S_n; $GL_n(\mathbb{R})$ etc.; products $G\times H$, $\mathbb{Z}^n$, ...

Lec. 2
- Like sets, groups can be finite ($\mathbb{Z}/n$, S_n, ...), countable ($\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Q}$, ...), uncountable ($\mathbb{R}$ ...)
- $H\subset G$ is a subgroup if $e\in H$, $a\in H\Rightarrow a^{-1}\in H$, $a,b\in H\Rightarrow ab\in H$. $|H|$ divides $|G|$.
  H, H' subgroups of $G\Rightarrow H\cap H'$ is a subgroup of G.
  All subgroups of $(\mathbb{Z},+)$ are $\mathbb{Z}n=\{mn\ /\ m\in\mathbb{Z}\}$ for some $n\geq 0$.
- A homomorphism $\varphi: G\to H$ is a map st. $\varphi(ab)=\varphi(a)\varphi(b)\ \forall a,b\in G$. ($\Rightarrow \varphi(a^{-1})=\varphi(a)^{-1}$)
  isomorphism = bijective homomorphism, automorphism = isom $G\xrightarrow{\sim}G$. $(\mathrm{Aut}(G),\circ)$ is a group.
- The kernel of $\varphi: G\to H$: $\mathrm{Ker}(\varphi)=\{g\in G\ /\ \varphi(g)=e_H\}$ subgroup of G. φ injective $\Leftrightarrow \mathrm{Ker}\,\varphi=\{e\}$
  The image of φ: $\mathrm{Im}(\varphi)=\{\varphi(g)\ /\ g\in G\}\subset H$ subgroup of H. φ surjective $\Leftrightarrow \mathrm{Im}\,\varphi=H$.

Lec. 3
- Given $a\in G$, $\varphi: \begin{matrix}\mathbb{Z}\to G\\ k\mapsto a^k\end{matrix}$ is a homomorphism with $\mathrm{im}(\varphi)=\langle a\rangle$ subgp generated by a.
  $\mathrm{Ker}(\varphi)=\mathbb{Z}n$ where $n=$ order of a $=\min\{n>0$ st. $a^n=e\}$.
  Hence the cyclic group $\langle a\rangle$ is $\simeq\mathbb{Z}/n$ if a has order n, $\simeq\mathbb{Z}$ if infinite order.
  ($a_1,\dots,a_k$ generate G if every element of G is a product of a_i and their inverses).
- A subgroup $H\subset G$ determines an equivalence relation (axioms: $a\sim a$; $a\sim b\Leftrightarrow b\sim a$; $\left.\begin{matrix}a\sim b\\ b\sim c\end{matrix}\right\}\Rightarrow a\sim c$)
  $a\sim b$ iff $a^{-1}b\in H$, whose equivalence classes are the (left) cosets $aH=\{ah\ /\ h\in H\}$.
  The quotient set: $G/H=\{\text{cosets } aH\}$. The index of H: $(G:H)=|G/H|=\dfrac{|G|}{|H|}$ if finite.

Lec. 4
- If G is finite: $H\subset G$ subgroup $\Rightarrow |H|$ divides $|G|$; $a\in G\Rightarrow \mathrm{ord}(a)\,|\,|G|$; $|G|=p$ prime $\Rightarrow G\simeq\mathbb{Z}/p$.
- A subgroup $H\subset G$ is normal $\Leftrightarrow \underset{\text{(left cosets = right cosets)}}{aH=Ha\ \forall a\in G} \Leftrightarrow \underset{\text{(conjugate subgroups)}}{aHa^{-1}=H\ \forall a\in G}$.
- The operation $(aH).(bH)=abH$ makes G/H a group iff H is a normal subgroup.
- $\forall\varphi: G\to H$ homomorphism, $\mathrm{Ker}(\varphi)=K$ is a normal subgroup of G, and $\mathrm{Im}(\varphi)\simeq G/K$.
  If φ is surjective, we have an exact sequence $\{1\}\to K\underset{\text{injective}}{\xrightarrow{i}} G\underset{\text{surjective}}{\xrightarrow{\varphi}} H\to\{1\}$ $\mathrm{Im}(i)=\mathrm{ker}(\varphi)$.
  Ex: $\{1\}\to \underset{\text{normal}}{H}\hookrightarrow G\to G/H\to\{1\}$; $0\to\mathbb{Z}/m\to\mathbb{Z}/mn\to\mathbb{Z}/n\to 0$ ($\mathbb{Z}/mn\simeq\mathbb{Z}/m\times\mathbb{Z}/n$ iff $\gcd(m,n)=1$)
  $\{e\}\to\mathbb{Z}/3\to S_3\xrightarrow{\text{sign}}\mathbb{Z}/2\to\{e\}$
  A homomorphism $G\xrightarrow{\varphi}H$ factors through $G\to G/K\xrightarrow{\bar\varphi}H$ iff $K\subset\mathrm{ker}\,\varphi$
- G is simple if its only normal subgroups are $\{e\}$ and itself. Ex: $\mathbb{Z}/p$ p prime; A_n $n\geq 5$.

Lec. 5
- Ex: the center $Z(G)=\{z\in G\ /\ zg=gz\ \forall g\in G\}$ is a normal subgroup (abelian: $zz'=z'z$)
- Ex: the commutator subgroup $[G,G]=\{\prod_{\text{finite}}[a_i,b_i]\}$, where $[a,b]=aba^{-1}b^{-1}$, is normal, and $G/[G,G]=\mathrm{Ab}(G)$ (abelianization) largest abelian quotient of G. $\forall G\xrightarrow{\varphi}H$, H abelian factors $G\to\mathrm{Ab}(G)\xrightarrow{\bar\varphi}H$.

Lec. 20 Artin 14.7
- Every finitely generated abelian group is $\simeq\mathbb{Z}^r\times\mathbb{Z}/n_1\times\dots\times\mathbb{Z}/n_k$ for some $r, n_1,\dots,n_k$.

Artin 6.7–6.12

- Group actions: G-action on set S: $G\times S\to S$, $(g,s)\mapsto g\cdot s$ st. $e.s=s\ \forall s\in S$, $(gh).s=g.(h.s)$ ($\Leftrightarrow$ homom. $\rho: G\to \mathrm{Perm}(S)$) ②

faithful if ρ injective; transitive if $\forall s,t\in S\ \exists g$ st. $gs=t$ (ie: 1 orbit)

Lec.21

- The orbit of $s\in S$ is $O_s = G\cdot s = \{g\cdot s \mid g\in G\}$. These form a partition $S=\coprod$ orbits.

The stabilizer of s is $\mathrm{Stab}(s)=\{g\in G \mid g\cdot s = s\}$ subgroup of G.

Elements in same orbit have conjugate stabilizer subgroups $\mathrm{Stab}(g\cdot s)=g\,\mathrm{Stab}(s)\,g^{-1}\subset G$.

Orbit-stabilizer: if $H=\mathrm{Stab}(s)$, then $G/H \simeq O_s$, $gH\mapsto g\cdot s$ bijection, in particular $|O_s|\cdot|\mathrm{Stab}\, s| = |G|$.

- Burnside's lemma (G, S finite): let $S^g=\{s\in S \mid gs=s\}$ fixed points of $g\in G$, then $\#\text{orbits} = \frac{1}{|G|}\sum_{g\in G}|S^g|$

Artin ch.7

- G acts on itself by left multiplication. This gives $G\hookrightarrow \mathrm{Perm}(G)$, hence: every finite group G is isomorphic to a subgroup of S_n, $n=|G|$.

- G acts on itself by conjugation: g acts by $h\mapsto ghg^{-1}$.

orbits = conjugacy classes; $\mathrm{Stab}(h)=\{g\in G \mid gh=hg\}=Z(h)$ centralizer of h.

Hence: $|G| = \sum_{\substack{C\subset G\\ \text{conj. classes}}} |C|$, where for each conj. class $|C_h| = \frac{|G|}{|Z(h)|}$ divides $|G|$. (class eq^n of G)

- For p-groups ($|G|=p^k$), the class equation $\Rightarrow |Z(G)|\geq p$ (number of conj classes of size 1)

Hence: • $|G|=p^2$, p prime $\Rightarrow G$ is abelian ($\simeq \mathbb{Z}/p\times\mathbb{Z}/p$ or $\mathbb{Z}/p^2$)

• 5 isom. classes of groups of order 8: $\mathbb{Z}/8$, $\mathbb{Z}/4\times\mathbb{Z}/2$, $(\mathbb{Z}/2)^3$, D_4, quaternion group.

Lec.22:

- $G\subset SO(3)$ finite subgp $\Rightarrow$ by considering the action of G on its poles (unit vectors along rotation axes), $G\simeq$ one of $\mathbb{Z}/n$, D_n (regular n-gon), A_4 (tetrahedron), S_4 (cube), A_5 (dodecahedron/icosahedron)

Lec.23:

- The symmetric group S_n is generated by transpositions (ij), in fact by $s_i=(i\ i{+}1)$.

- $\forall \sigma\in S_n\ \exists$ unique decomp of σ as product of disjoint cycles $(a_1\dots a_k)$.

$\sigma,\tau\in S_n$ are in same conjugacy class iff they have the same cycle lengths.

- the alternating group $A_n=\mathrm{Ker}(\mathrm{sign}: S_n\to\mathbb{Z}/2) = \{$products of even # of transpositions$\}$

Lec.24:

A conjugacy class in S_n which consists of even permutations is either 1 or 2 conj. classes in A_n; it splits into 2 iff the centralizer $Z(\sigma)\subset A_n$ ($\Leftrightarrow$ cycle lengths of σ are all odd & distinct).

- A_n is simple for $n\geq 5$ (A_4 isn't: $\{id, (ij)(kl)\}\simeq \mathbb{Z}/2\times\mathbb{Z}/2$ is normal in A_4 and S_4).

Lec.25:

- Sylow theorems: $|G|=p^e m$, $p\nmid m$ $\Rightarrow$ a Sylow p-subgroup of G is a subgp. of order p^e.

Thm 1: $\forall p$ prime $\mid |G|$, G contains a Sylow p-subgroup. ($\rightarrow$ consequence: G contains an elt of order p)

Thm 2: all Sylow p-subgroups of G are conjugates of each other, and every subgroup of order p^k ($k\leq e$) is contained in a Sylow subgroup.

Thm 3: the number s_p of Sylow p-subgroups satisfies $s_p\equiv 1 \bmod p$ and $s_p \mid m = \frac{|G|}{p^e}$.

- If G contains subgroups N, H st. $N\cap H=\{e\}$ (eg because $\gcd(|N|,|H|)=1$) and $|G|=|N|\cdot|H|$, then $\forall g\in G\ \exists$ unique $n\in N$, $h\in H$ st. $g=nh$.

If N and H are both normal in G then $G\simeq N\times H$. If N is normal but not H, we have a semidirect product $N\rtimes_\varphi H$, $\varphi: H\to \mathrm{Aut}(N)$ given by conjugation inside G.

$$(n,h)\cdot(n',h') = (n\,\varphi(h)(n'),\ hh')$$

Lec. 26 • given $H \subset G$ (eg. p-Sylow), the number of conjugate subgroups $gHg^{-1} \subset G$ (eg. all p-Sylows) equals $|G/N(H)|$, $N(H)$ normalizer $= \{g \in G \mid gHg^{-1} = H\}$ (largest subgp of G st. H is normal inside N). ③

• Example: $|G| = 15 \Rightarrow$ Sylow subgroups of order 3 and 5 are normal $(s_3 = s_5 = 1) \Rightarrow G \simeq \mathbb{Z}/3 \times \mathbb{Z}/5$.

$|G| = 21 \Rightarrow s_3 \in \{1, 7\}$, $s_7 = 1$, so either $G \simeq \mathbb{Z}/3 \times \mathbb{Z}/7$ or semidirect product $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$.

Lec. 27 $|G| = 12 \Rightarrow$ 1 or 3 2-Sylows, 1 or 4 3-Sylows, one of these is normal $\Rightarrow$ 5 isom. classes:

$\mathbb{Z}/4 \times \mathbb{Z}/3$, $(\mathbb{Z}/2)^2 \times \mathbb{Z}/3$, A_4, D_6, $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$.

• The free group $F_n = \langle a_1 \dots a_n \rangle = \{$all reduced words $a_{i_1}^{m_1} \dots a_{i_k}^{m_k}\}$ (words in $a_i^{\pm 1}$ never simplify except $a_i a_i^{-1} = a_i^{-1} a_i = 1$)

• Any group G with n generators $g_1 \dots g_n$ is a quotient of F_n, via $\varphi: F_n \twoheadrightarrow G$, $a_i \mapsto g_i$

G is finitely presented if $\mathrm{Ker}(\varphi)$ is generated by a finite set $r_1, \dots, r_k$ & their conjugates

Write $G \simeq \langle g_1 \dots g_n \mid r_1 \dots r_k \rangle = F_n / \langle$normal subgp gen^d by conjugates of $r_j\rangle$.

• The Cayley graph of G w/ generators g_i: vertices = elements of G

edges: connect g to $g \cdot g_i$ $\forall g \in G$, $\forall g_i$

• A normal form for elements of $G = \langle g_1 \dots g_n \mid r_1 \dots r_k \rangle$ is a set of words in $g_1^{\pm 1} \dots g_n^{\pm 1}$ st. every element of G appears exactly once among these.

Lec. 28 • Ex: $S_n \simeq \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1,\ s_i s_j = s_j s_i\ |i-j| \geq 2,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$.

$SL_2(\mathbb{Z})$ is gen^d by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\} = \langle S, T \mid S^2, (ST)^3 \rangle$

Lec. 29 • A representation of G is a vector space V on which G acts by linear operators; ie. $\rho: G \to GL(V)$ homomorphism.

Artin ch. 10, Fulton-Harris ch. 1–2 • A subrepresentation is a subspace $W \subset V$ invariant under G: $g(W) = W$ $\forall g \in G$.

V is irreducible if has no nontrivial subrepresentations

• G finite, V finite dim./$\mathbb{C}$: each $g: V \to V$ has finite order, $g^n = Id \Rightarrow$ diagonalizable, $\lambda_j = e^{2\pi i k/n}$

• if G is abelian, all operators $g: V \to V$ are simultaneously diagonalizable $\Rightarrow$ irred. rep^s are 1-dim^l. These correspond to elements of the dual group $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$. (Note $\widehat{\mathbb{Z}/m}$ is $\simeq \mathbb{Z}/m$)

• a homomorphism of representations is a G-equivariant linear map, ie. $\varphi(gv) = g\varphi(v)$.

• V, W rep^s of $G \Rightarrow$ so are $V \oplus W$, $V \otimes W$ $(g: v \otimes w \mapsto gv \otimes gw)$, V^* $(\ell \mapsto \ell \circ g^{-1})$,

$V^* \otimes W \simeq \mathrm{Hom}(V, W)$ $(\varphi \mapsto g \circ \varphi \circ g^{-1})$. $(\mathrm{Hom}_G(V, W) =$ invariant part $\mathrm{Hom}(V, W)^G)$

Lec. 30 • Any $\mathbb{C}$-representation of a finite group G admits an invariant Hermitian inner product, with respect to which G acts by unitary operators.

• V rep. of a finite group (over $\mathbb{C}$), $W \subset V$ invariant subspace $\Rightarrow \exists U \subset V$ invariant st. $V = U \oplus W$.

Hence: any $\mathbb{C}$-representation of a finite group decomposes into a direct sum of irreducibles.

• Schur's lemma: V, W irred. rep's of $G \Rightarrow$ any homom. $\varphi \in \mathrm{Hom}_G(V, W)$ is either zero or an isomorphism; and all iso's of an irred. rep. are multiples of id: $\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot id_V$

• Ex: reps. of S_n: trivial rep $U = \mathbb{C}$, σ acts by id; alternating rep: $U' = \mathbb{C}$, σ acts by $(-1)^\sigma$.

standard rep. (dim. n-1): $V = \{(z_1, \dots, z_n) \mid \Sigma z_i = 0\} \subset \mathbb{C}^n$, σ acts by permuting coords: $e_i \mapsto e_{\sigma(i)}$.

U, U', V are the only irred. reps of S_3.

Lec. 31 : • The key tool to study representations is the <u>character</u> $\chi_V : G \to \mathbb{C}$, $\chi_V(g) = tr(g : V \to V)$
(In terms of eigenvalues, $tr(g) = \sum \lambda_i$, and $tr(g^k) = \sum \lambda_i^k$, so χ_V recovers all symmetric polynomial expressions in the λ_i, hence the λ_i as unordered tuple).
$\chi_V : G \to \mathbb{C}$ is a class function, ie. $\chi_V(hgh^{-1}) = \chi_V(g)$.

• $\chi_{V \oplus W} = \chi_V + \chi_W$, $\chi_{V \otimes W} = \chi_V \chi_W$, $\chi_{V^*} = \overline{\chi_V}$, $\chi_{Hom(V,W)} = \overline{\chi_V}\chi_W$.

• for a permutation rep. (G acting on $S \rightsquigarrow G$ acts on V with basis $(e_s)_{s\in S}$, $g \cdot e_s = e_{g\cdot s}$)
$\chi(g) = \#\{s \in S / g \cdot s = s\} = |S^g|$.

Lec. 32 • Character table of G = list, for each irred. rep. V_i, the value of χ_{V_i} on each conjugacy class.

• $\varphi = \frac{1}{|G|}\sum_{g\in G} g : V \to V$ projection onto $V^G = \{v \in V / gv = v\ \forall g\}$, so $\dim(V^G) = tr(\varphi) = \frac{1}{|G|}\sum_g \chi_V(g)$

• $H(\alpha, \beta) = \frac{1}{|G|}\sum_{g\in G} \overline{\alpha}(g)\beta(g)$ Hermitian inner product on $\mathbb{C}_{class}(G) = \{$class functions $G \to \mathbb{C}\}$
then $\dim Hom_G(V, W) = H(\chi_V, \chi_W)$.

• The characters of the irreducible reps of G are an <u>orthonormal basis</u> of $(\mathbb{C}_{class}(G), H)$.
In particular the number of irred. reps = number of conjugacy classes

• The multiplicities a_i in the decomposition of a G-rep. W into irreducibles $W \simeq \bigoplus_i V_i^{\oplus a_i}$
are given by $a_i = \dim Hom_G(V_i, W) = H(\chi_{V_i}, \chi_W)$. Moreover, $H(\chi_W, \chi_W) = \sum a_i^2$.

• The regular rep^n of G (= permutation rep. for G acting on itself by left multiplication)
contains each irred. rep. V_i with multiplicity = $\dim V_i$; therefore $|G| = \sum_i (\dim V_i)^2$.

Lec-33-34 • These results allow us to find character tables of various groups (eg. S_4, A_4, S_5, A_5) by starting from known representations, considering tensor products, and using $H(\cdot,\cdot)$ pairings and orthogonality to find irreducible pieces & the missing irreducible reps.