<u>Real representations</u>: we've studied actions of finite groups on complex vector spaces, now we want to do the same for real ones.

- Existence of an invariant inner product still holds (build $\langle \cdot,\cdot \rangle$ by averaging).
  $\Rightarrow$ every rep. is $\oplus$ of irreducibles (given a subrep., its $\perp$ is also a subrep.)
- Schur's lemma fails: $\mathbb{Z}/n$ acts on $\mathbb{R}^2$ by rotations, this is irreducible, has nontrivial automorphisms eg any rotation of $\mathbb{R}^2$

Main tool to study real rep^{ns}: <u>complexification</u>

$\{\text{real rep}^{ns}\} \longrightarrow \{\text{complex rep}^{ns}\}$

$$V_0 \longmapsto V = V_0 \otimes_{\mathbb{R}} \mathbb{C} = V_0 \oplus i V_0. \quad (G \text{ acts by } g(v+iw) = gv + i\, gw).$$

<u>Def</u>: A complex rep. V of G is called <u>real</u> if there exists a rep. over $\mathbb{R}$, V_0, st. $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$

- Necessary condition: χ_V must take real values! This is alas <u>not</u> a sufficient cond^{n}
  (trace of matrix with $\mathbb{R}$ entries is real!)

<u>Ex</u>: the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, $i^2 = j^2 = k^2 = ijk = -1$ acts faithfully on $\mathbb{C}^2$ by

$$\pm 1 \mapsto \pm \mathrm{Id}, \quad \pm i \mapsto \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \pm j \mapsto \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm k \mapsto \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$\chi(\pm 1) = \pm 2$, all others have $\chi = 0$: so χ takes real values.

However this does not come from a 2-dimensional real representation: $Q \not\hookrightarrow GL(2,\mathbb{R})$.
(This is because a real representation of a finite group has an invariant inner product, by the same averaging trick as in the complex case, so we'd get $Q \hookrightarrow O(2)$ (since faithful), with -1 acting by $-\mathrm{Id}$, but only 2 elements of $O(2)$ square to $-\mathrm{Id}$ (rotations by $\pm 90°$) while we need 6 such elements for $\pm i, \pm j, \pm k$.)

<u>Prop</u>: A complex representation V is real iff there exists a G-equivariant, complex antilinear map $\tau : V \to V$ (ie. $\tau(\lambda v) = \bar{\lambda}\tau(v)$) such that $\tau^2 = id$.

<u>Pf</u>:
- $\Rightarrow$ is clear: if $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, let $\tau(v+iw) = v - iw$ for $v, w \in V_0$: complex conjugation!
- $\Leftarrow$: given τ, $v \in V$ decomposes into $Re(v) = \dfrac{v + \tau(v)}{2}$ and $i\,Im(v) = \dfrac{v - \tau(v)}{2}$ which belong to the ± 1 eigenspaces of τ. Let $V_0 = \ker(\tau - id)$, which is an $\mathbb{R}$-subspace of V (not a $\mathbb{C}$-subspace!) and, as $\mathbb{R}$-linear maps, $\tau i = -i\tau$ so iV_0 is the -1-eigenspace, and $V = V_0 \oplus iV_0 \simeq V_0 \otimes_{\mathbb{R}} \mathbb{C}$. The above was just linear algebra, but G-equivariance of τ implies that the eigenspace $V_0 = \ker(\tau - 1)$ is preserved by G, hence a subrep. over $\mathbb{R}$ (similarly for iV_0). $\square$

- If a complex rep. V is real, then G acts by real matrices $\Rightarrow \chi_V$ takes values in $\mathbb{R}$

Conversely, let V be an irreducible complex rep. of G, such that χ_V takes values in $\mathbb{R}$. 2

(⚠ even if V_0 is irreducible / $\mathbb{R}$, $V_0 \otimes_{\mathbb{R}} \mathbb{C}$ might not be irreducible / $\mathbb{C}$, see eg. $\mathbb{Z}/n \underset{\text{rotations}}{\subset} \mathbb{R}^2$).

Then $\chi_V = \overline{\chi_V} = \chi_{V^*}$, so $V \simeq V^*$ as G-reps.

Recall: a linear map $\varphi : V \to V^*$ determines a bilinear form $B : V \times V \to \mathbb{C}$, $B(v,w) = \varphi(v)(w)$.
B is G-invariant iff φ is G-equivariant. Thus, Schur's lemma for $V \simeq V^*$ irreducible
$\Rightarrow$ V admits a G-invariant bilinear form B, unique up to scaling, and nondeg. if nonzero.

Now, recall $B \in (V \otimes V)^* = \mathrm{Sym}^2 V^* \oplus \Lambda^2 V^*$, ie. the symmetric and skew parts of B $\left(= \frac{1}{2}(B(v,w) \pm B(w,v))\right)$ are also G-invariant bilinear forms on V. By uniqueness, one of these is zero and the other is nondegenerate; ie. B is either symmetric or skew.

The symmetric case corresponds to real rep$^{\text{ns}}$; the skew-symm. case gives "quaternionic" rep$^{\text{s}}$

Prop: An irreducible complex representation V of a finite group G is real iff V carries a G-invariant nondegenerate symmetric bilinear form $B : V \times V \to \mathbb{C}$.

Pf:
- Assume $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ is real. Then V_0 has an invariant real inner product B; extend $\mathbb{C}$-bilinearly: $B(v_1 + i w_1, v_2 + i w_2) := B(v_1, v_2) + i B(w_1, v_2) + i B(v_1, w_1) - B(v_2, w_2)$.
defines a nondegenerate symmetric bilinear form on V.
- Conversely: $B : V \times V \to \mathbb{C}$ determines an isom. $\varphi : V \to V^*$ ($\mathbb{C}$-linear, equivariant); choosing an invariant Hermitian inner product H on V, we also have a $\mathbb{C}$-antilinear equivariant bijection $V \to V^*$. Composing one with the inverse of the other gives a $\mathbb{C}$-antilinear equivariant map $\tau : V \to V$, characterized by: $H(\tau(v), w) = B(v,w)$.
τ^2 is now an equivariant $\mathbb{C}$-linear isom. $V \to V$, hence $\tau^2 = \lambda\,\mathrm{Id}$ by Schur.
A calculation: $H(\tau^2(v), v) = B(\tau(v), v) = B(v, \tau(v)) = H(\tau(v), \tau(v)) \geq 0$
shows $\lambda \in \mathbb{R}_+$; replacing H by $\lambda^{1/2} H$ we can arrange $\tau^2 = id$.
Thus V is real by the previous prop. □

* In the other case where the invariant bilinear form B is skew-symmetric, the same argument gives a $\mathbb{C}$-antilinear equivariant bijective map $J : V \to V$ which now satisfies $J^2 = -id$.
This is a *quaternionic* structure on V, ie. describes a structure of $\mathbb{H}$-module on V where $\mathbb{H}$ = quaternions $= \{a + bi + cj + dk \,/\, a,b,c,d \in \mathbb{R}\}$ $\quad i^2 = j^2 = k^2 = ijk = -1$

"division algebra" (noncommutative analogue of a field: $\mathbb{H}$ is a noncommutative ring st. every nonzero element has a multiplicative inverse). $\mathbb{H} = \mathbb{C}1 \oplus \mathbb{C}j$, with $ji = -ij$, $j^2 = -1$, so an $\mathbb{H}$-module is the same thing as a $\mathbb{C}$-vector space + antilinear map j st. $j^2 = -id$.

Ex: the regular rep. V of S_3 is real. This can be seen directly if we notice that $S_3 \simeq D_3$ acts on $V_0 = \mathbb{R}^2$ by rotations and reflections, and $V_0 \otimes_{\mathbb{R}} \mathbb{C} \simeq V$... or more abstractly by observing $V^* \simeq V$, and $\Lambda^2 V^* \simeq U'$ has no trivial summand hence $\nexists$ invariant skew-symmetric $B \in \Lambda^2 V^*$, but $\mathrm{Sym}^2 V^* \simeq U \oplus V$ has a trivial summand giving an invariant symmetric bilinear form $B \in \mathrm{Sym}^2 V^*$ & applying the above.

Ex: the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, $i^2 = j^2 = k^2 = ijk = -1$ acts on $\mathbb{C}^2$ by

$$\pm 1 \mapsto \pm \mathrm{Id}, \quad \pm i \mapsto \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \pm j \mapsto \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm k \mapsto \mp\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

χ takes real values, but this doesn't come from a real representation: $Q \not\hookrightarrow O(\mathbb{R}^2)$.
Rather, this is a quaternionic representation: $\mathbb{H} \simeq \mathbb{C} \oplus j\mathbb{C}$, the above linear maps correspond to left-multiplication by elements of Q. (eg: $i(z_1 + jz_2) = iz_1 + j(-iz_2)$
$k(z_1 + jz_2) = (-iz_2) + j(-iz_1)$).
The $\mathbb{C}$-antilinear map $J: V \to V$, $J^2 = -1$ is right multiplication by j (commutes with left-mult. V)

We'll end here with the content on representation theory.. What comes next in math?

- Within algebra, the recommended next topic to study is rings, modules, fields.
  This is Math 123 (offered every year; this spring taught by Prof. Mark Kisin)
  Independently, you could explore some number theory (Math 124 (F.) easier / 129 (S.) harder)
  After 123 you could look at alg. geometry (Math 137), or jump to graduate level algebra (start with Math 221 if you've only taken 123).
  (Combinatorics - Math 155r is also a possibility if you want something more fun).
- but... at this point the recommended thing to do this spring is study analysis (& topology).
  Math 55b covers some real analysis fairly quickly, but also goes over a good amount of topology (Math 131) and complex analysis (Math 113). The material has no logical dependency on 55a (except maybe def$^{\underline{n}}$ of a group and a vector space).
  On the other hand the pace, the workload, and the people you'll interact with are mostly the same as in 55a. If you are tired of the pace (or of the people), 25b is a completely reasonable choice too (Math 131 and 113 can be taken separately later on!), and, if this fits your learning style or schedule better, it does not affect in any way your trajectory towards math graduate programs if that's your goal.
- At some point you should consider declaring a math concentration!

Also, if the end goal is a math PhD, look into research opportunities.

→ There are some on-campus, but more likely to work out after you've taken some more specialized math classes. (also, Office of Undergrad Research & Fellowships has $$ for summer research on campus if you have found someone to work with).

→ Better: look into REUs Research Experiences for Undergraduates (list at NSF) (most are for US citizens/residents only). Some are more prestigious/competitive than others; some have more prerequisites, or specialized topics, but many of them should be perfectly accessible to you after Math 55. Applications due by February.

https://www.math.harvard.edu/undergraduate/undergraduate-research/
https://www.nsf.gov/crssprgm/reu/list_result.jsp?unitid=5044

But before that... final exam!

- The exam will be posted on Canvas on Monday December 6, and will be due on Canvas by Monday December 13. (Hopefully it won't take the whole week to complete! The goal is to give you flexibility in when you plan to work on it). The final will soon appear under "Assignments" on the course Canvas site (minus the actual exam, to appear 12/6).

- The basic format will be similar to the midterm (several problems, mostly multi-part, and of variable difficulty levels), but at a more ambitious scale -- there's more material covered, and your math skills have grown since early October. Importantly: I don't necessarily expect most of you to complete the whole exam. The goal of some of the more challenging questions is to see how you approach a problem, even if you are not able to get to a complete solution. On just one problem, progress on the further parts may depend strongly on part (a); if so this will be clearly stated, along with instructions to request a hint on part (a) if you are stuck. The material covered is what we've seen in class up to Lecture 34 (November 22) included.

- As with the midterm: no collaboration will be allowed; no materials other than lecture notes, and the textbooks we've used (Artin, Axler, Fulton-Harris) + handouts on tensors/....

- A two-part summary of the main concepts and results seen in class, in video form (alongside the lecture videos) and as handwritten notes (alongside the lecture notes), is on Canvas, as well as a selection of potential review problems from the textbooks.

- I am holding office hours ... today (Wed 12/1) 12:15-1:15 in 411
  + Friday and Monday 12/6 10:30 – 12noon (in 507 if available?)
  (exam will be posted after that)

  See Slack for CA office hours announcements.
- Feel free to email (or ask on Slack; I check email more regularly) w/ any questions.

ANY QUESTIONS ?

PLEASE COMPLETE OFFICIAL COURSE EVALUATIONS

Unlike the Canvas surveys, these actually get seen by
- future students
- my colleagues & the university

and influence the planning & staffing of math courses in future semesters!