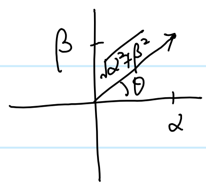
Bode Plot and Block Diagram

Reading: Astrom and Murray, Chapter 8;
Additional reference:
Franklin, Powell, Emami-Naeini, Feedback Control of Dynamic Systems, section 3.1, section 3.2, section 6.1.1

Bode Plot:

- Evaluate transfer function for $s = iw$, $w \geq 0$
- $G(iw)$ is just a complex number

$$\alpha + \beta i = \sqrt{\alpha^2+\beta^2}\; e^{i\theta}, \quad \theta = \tan^{-1} \beta/\alpha$$

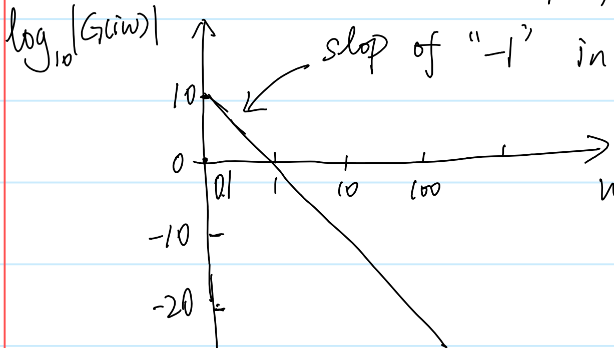


- E.g. $G(s) = \frac{1}{s} \Rightarrow G(iw) = \frac{1}{iw}$

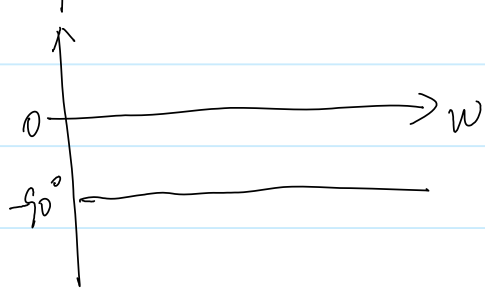
Magnitude: $|G(iw)| = \frac{1}{w}$, $\log_{10}|G(iw)| = \log_{10} 1 - \log_{10} w = -\log_{10} w$

Phase : $\angle G(iw) = -90°$,

slop of "−1" in log-log scale.



(in some practice, plot in $\underline{20dB}$ / dB scale.
$20\log_{10}|G(iw)|$ $\log_{10} w$.



- Magnitude multiply

Phases add

$$(\alpha_1+\beta_1 i)(\alpha_2+\beta_2 i) = \sqrt{\alpha_1^2+\beta_1^2}\ \sqrt{\alpha_2^2+\beta_2^2}\ e^{i(\theta_1+\theta_2)}$$

Note that for any polynomial it can be written as $c(s+z_1)(s+z_2)\cdots(s+z_n)$ (coiefficient)

$z_i \in \mathbb{C}$ (roots of the polynomial)

$$G(s) = c\cdot\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)};$$

$-z_i$: zeros

$-p_i$: poles

Then: $$G(i\omega) = \frac{N_1 N_2\cdots N_m}{D_1 D_2\cdots D_n}\, e^{i[(\theta_1+\theta_2+\cdots+\theta_m)-(\phi_1+\phi_2+\cdots+\phi_n)]}$$

where $N_i = \sqrt{w_i^2+z_i^2}$, $\theta_i = \tan^{-1}(w/z_i)$

$D_i = \sqrt{w_i^2+p_i^2}$, $\phi_i = \tan^{-1}(w/p_i)$

- Examples:
  - i) $s^{\pm n}$
  - ii) $\frac{1}{s+a}$, $a>0$, $\frac{1}{s+a}$ $a<0$ (unstable)
  - iii) $\frac{s+b}{s+a}$, $a \ll b$ or $b \ll a$
  - iv) $\frac{1}{s^2+2\zeta w_n s+w_n^2}$
  - v) $e^{-s\tau}$ (Time delay): If $y(t) = u(t-\tau)$, $Y(s) = e^{-s\tau}U(s)$
  - vi) $s+a$ $a<0$ (RHP zero)

Useful matlab commands:
sys=ss(A,B,C,D)
G=tf(sys);
Or: G=ss2tf(A,B,C,D);
Or: n=[0 0 1]; d=[1 0.1 1], G=tf(n,d); % This defines G=1/(s^2+0.1*s+1)
Or: s=tf('s'); G=1/(s^2+0.1*s+1);
pzmap(G): plots the poles and zeros of G
bode(G): Bode plot of G
Try those matlab codes on the above examples and we will go through them in details in the next class.

Sketch of Bode plot (examples)

Sketch of Bode plot (examples).

1) s^k, $k>0$.

$\log|G(iw)| = \log(w^k) = k \log w$

$\angle G(iw) = 90k$.

- $k=1$: gain curve is a straight line with slope $k=1$
  phase curve is a constant at $90^\circ \times k = 90^\circ$
- $k=-1$: gain curve is a straight line with slope $k=-1$
  phase curve is a constant at $90^\circ \times k = -90^\circ$.

Figure 8.12.

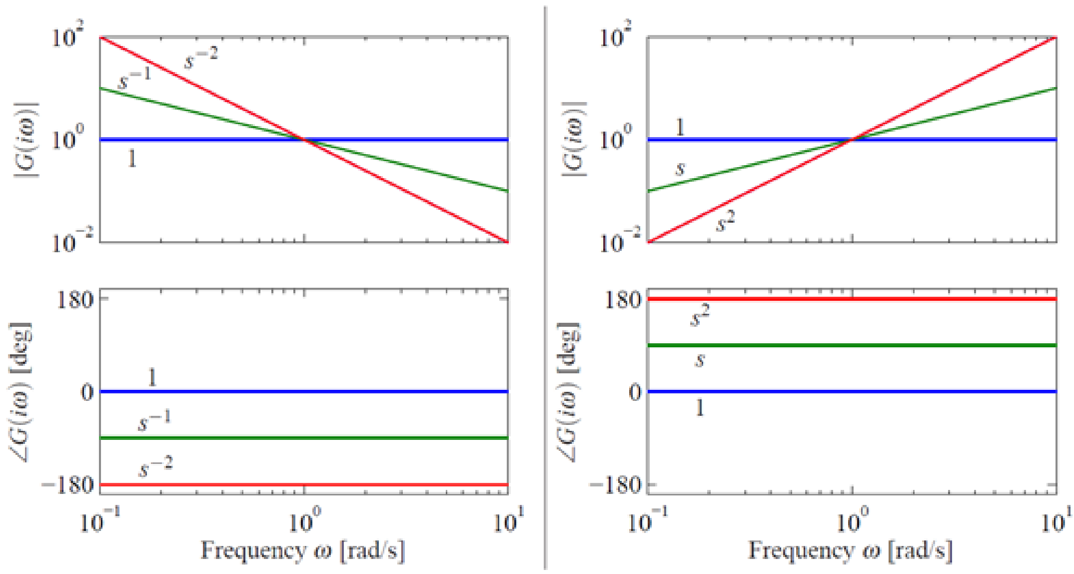


Figure 8.12: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$. On a log-log scale, the gain curve is a straight line with slope k. Using a log-linear scale, the phase curves for the transfer functions are constants, with phase equal to $90^\circ \times k$

2). $G(s) = \dfrac{a}{s+a}$ $\quad a>0$

$|G(iw)| = \dfrac{|a|}{|iw+a|}$, $\quad \angle G(iw) = \angle a - \angle(iw+a)$
$= -\frac{180}{\pi} \arctan w/a$

$\log|G(iw)| = \log a - \frac{1}{2}\log(w^2+a^2)$

Note:

$$\log|G(iw)| = \log a - \frac{1}{2}\log(w^2+a^2)$$

Note:

$$\Rightarrow \log|G(iw)| \approx \begin{cases} 0 & \text{if } w<a \quad (\log(w^2+a^2) \approx \log a^2) \\ \log a - \log w & \text{if } w>a \quad (\log(w^2+a^2) \approx \log w^2) \end{cases}$$

$$\angle G(iw) \approx \begin{cases} 0 & \text{if } w < a/10 \\ -45-45(\log w - \log a) & \text{if } a/10 < w < 10a \\ -90 & \text{if } w > 10a. \end{cases}$$

Gain Curve: break point is a; before the break point, staright line with slope 0; after the break point, straight line with slope -1.

Phase Curve: Phase curve is zero up to frequency $a/10$ and the decreases linearly by $45°/dB$ up frequency $10a$, at which point it remains constant at $90°$.

(Notice that a first order system behaves like a constant for low frequencies, and like an integrator for high frequencies. (compare Bode plots of s^k))

Bode plots: Figure 8.13.

3). $G(s) = \dfrac{w_0^2}{s^2+2w_0\zeta s + w_0^2}$ (Complex poles)

$$\log|G(iw)| = 2\log w_0 - \frac{1}{2}\log(w^4 + 2w_0^2 w^2(2\zeta^2-1) + w_0^4)$$

$$\angle G(iw) = -\frac{180}{\pi}\arctan\frac{2\zeta w_0 w}{w_0^2 - w^2}$$

$$\log|G(iw)| \approx \begin{cases} 0 & \text{if } w \ll w_0 \\ 2\log w_0 - 2\log w & \text{if } w \gg w_0 \end{cases}$$

$$\angle G(iw) \approx \begin{cases} 0 & \text{if } w \ll w_0 \\ -180 & \end{cases}$$

ω if $w \gg w_0$

$\max_w |G(iw)| \approx 1/(2\zeta)$ for $w \approx w_0$ (called Q-curve).

Gain Curve: has an asymptote with zero slope for $w \ll w_0$

has an asymptote with slope -2

peak value at $w \approx w_0$.

Phase Curve: is zero for low frequencies

approches 180° for large frequencies.

Bode Plot is in Figure 8.13.

(More examples and principles on Section this week!)

(You will be asked to sketch bode plots for simple cases).

(Additional Note is posted on Website for Bode plots).

- Magnitude drops for a pole

increases for a zero.

change more dramastically for higher order poles/zeros.

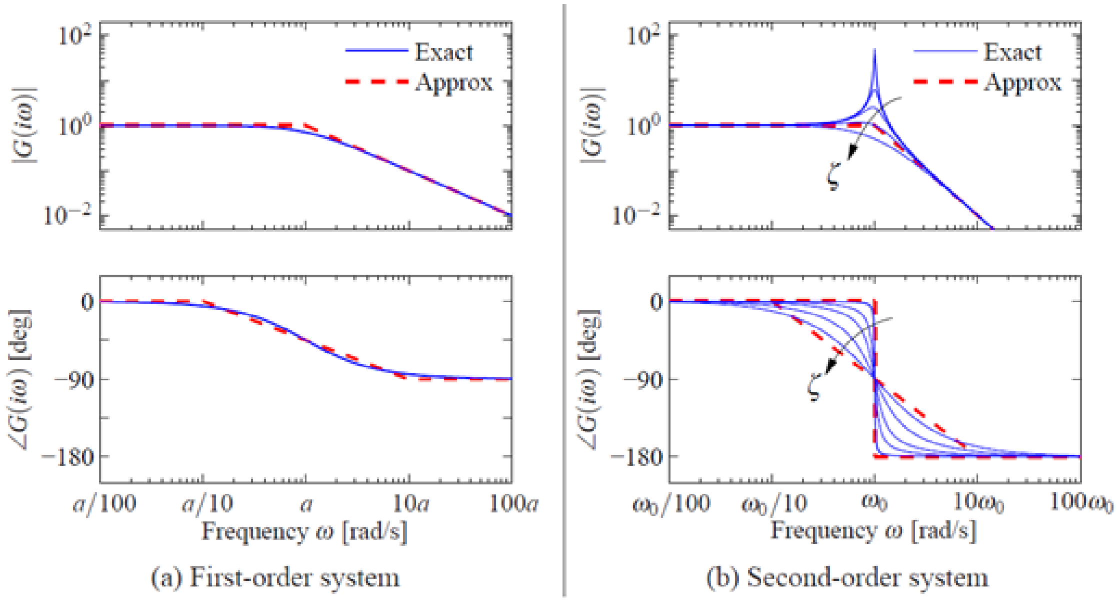


Figure 8.13: Bode plots for first- and second-order systems. (a) The first-order system $G(s) = a/(s+a)$ can be approximated by asymptotic curves (dashed) in both the gain and the frequency, with the breakpoint in the gain curve at $\omega = a$ and the phase decreasing by 90° over a factor of 100 in frequency. (b) The second-order system $G(s) = \omega_0^2/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$ has a peak at frequency a and then a slope of -2 beyond the peak; the phase decreases from 0° to −180°. The height of the peak and the rate of change of phase depending on the damping ratio ζ ($\zeta = 0.02$, 0.1, 0.2, 0.5 and 1.0 shown).

– Low / Band / High Pass Filter.

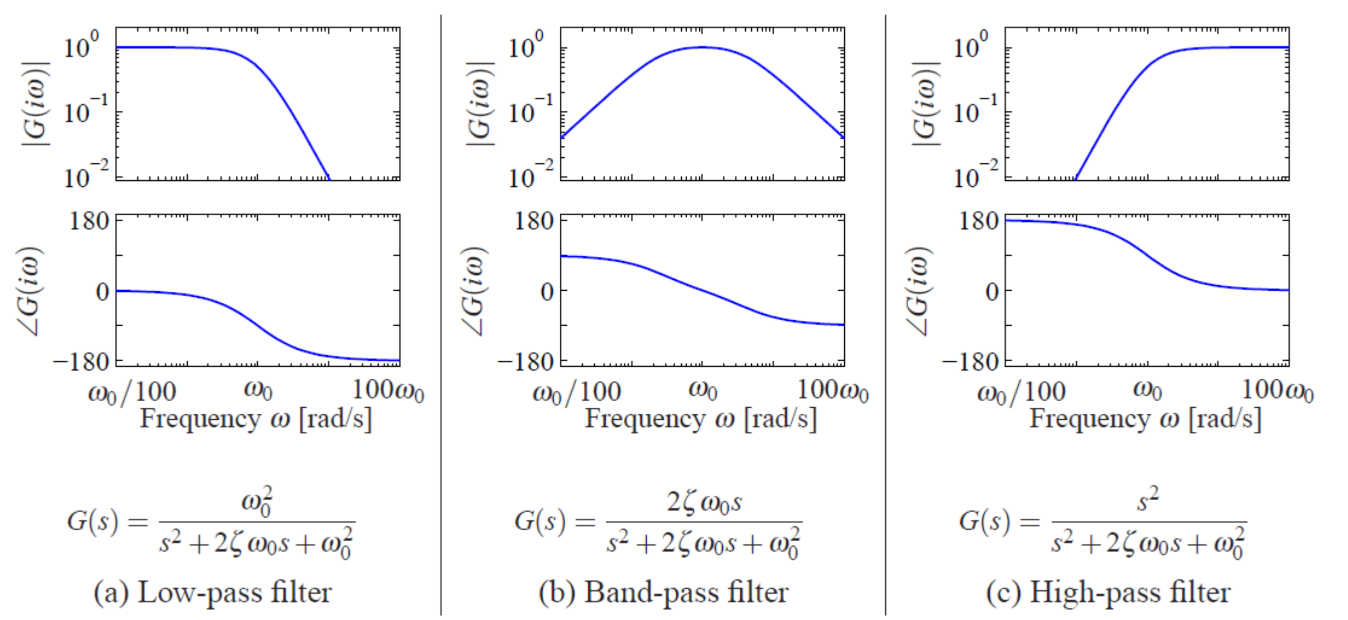


Figure 8.15: Bode plots for low-pass, band-pass and high-pass filters. The top plots are the gain curves and the bottom plots are the phase curves. Each system passes frequencies in a different range and attenuates frequencies outside of that range.

Series Interconnections

Q: what happens when we connect two systems together *in series?*

$u_1 \rightarrow$

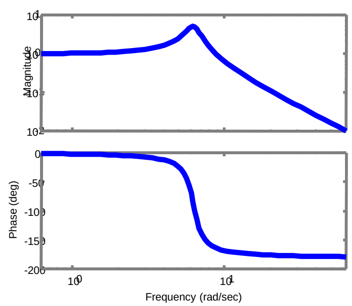
$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$y_1 = C_1 x_1 + D_1 u_1$$

$\xrightarrow{y_1 \qquad u_2}$

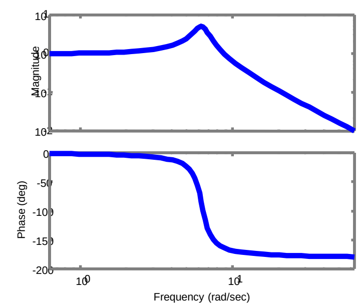
$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$y_2 = C_2 x_2 + D_2 u_2$$

$\xrightarrow{y_2}$

$u = A\sin(\omega t) \rightarrow$



$y = g_1 A \sin(\omega t + \phi_1) \rightarrow$



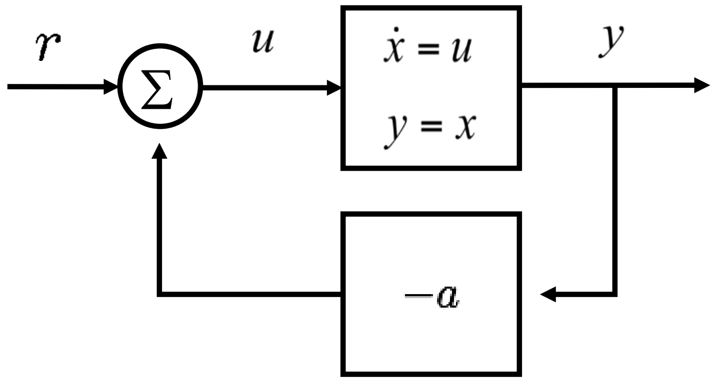
$y_2 = g_1 g_2 A \times \sin(\omega t + \phi_1 + \phi_2) \rightarrow$

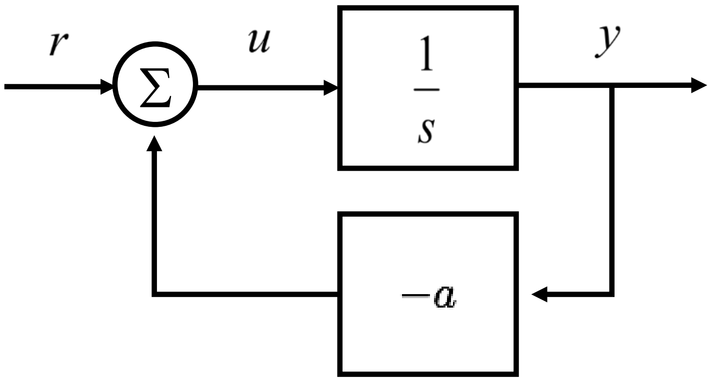
- A: Transfer functions *multiply*
  - Gains multiply
  - Phases add
  - Generally: transfer functions well formulated for frequency domain interconnections

$u_1 \rightarrow \boxed{G_1(s)} \xrightarrow{u_2 = y_1} \boxed{G_2(s)} \xrightarrow{y_2}$

$u_1 \rightarrow \boxed{G_2(s)G_1(s)} \rightarrow y_2 = G_2(s)G_1(s)u_1$

Feedback Interconnection





- State space derivation

$$\dot{x} = u = r - ay = -ax + r$$
$$y = x$$

- Frequency response: $r = A\sin(\omega t)$

$$y = \left|\frac{1}{\sqrt{a^2 + \omega^2}}\right| \sin\left(\omega t - \tan^{-1}(\frac{\omega}{a})\right)$$

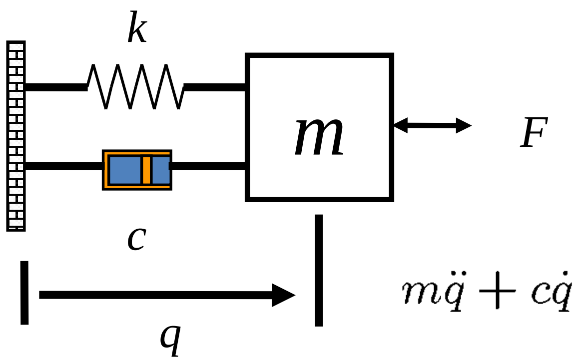
- Transfer function derivation

$$y = \frac{u}{s} = \frac{r - ay}{s}$$
$$y = \frac{r}{s + a} = G(s)r$$

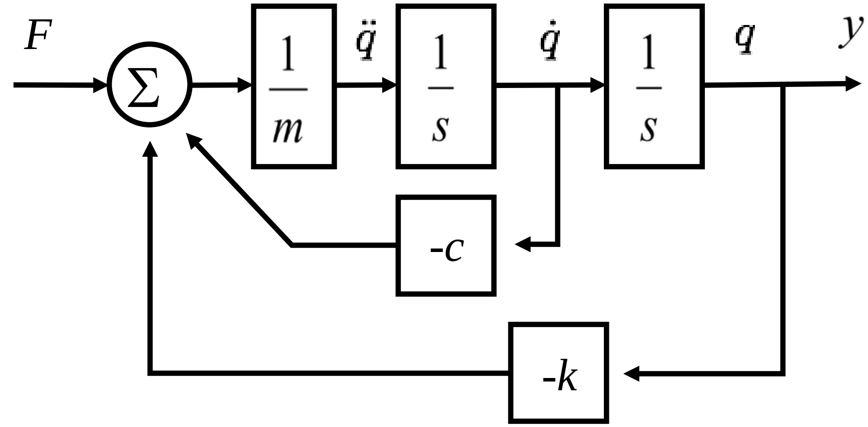
- Frequency response

$$y = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

Example: mass spring system



$$m\ddot{q} + c\dot{q} + kq = F$$



- Rewrite in terms of "block diagram"
  - Represent integration using 1/*s*
  - Include spring and damping through feedback terms
  - Determine the transfer function through algebraic manipulation
  - Claim: resulting transfer function captures the frequency response

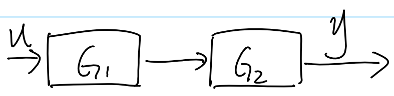
$$y = \frac{1}{m} \cdot \frac{1}{s} \cdot \frac{1}{s}(F - c\dot{q} - kq) = \frac{1}{ms^2}F - \frac{c}{ms}y - \frac{k}{ms^2}y$$

$$\left(1 + \frac{c}{ms} + \frac{k}{ms^2}\right) y = \frac{1}{ms^2}F$$

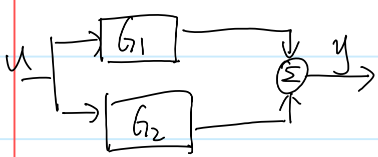
$$y = \frac{1}{ms^2 + cs + k}F$$

$$\boxed{H(s) = \frac{1}{ms^2 + cs + k}}$$

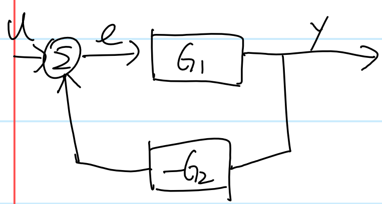
Block Diagram



: $G_{u\to y} = G_2 G_1$



: $G_{u\to y} = G_1 + G_2$



: $y = G_1 e$

$e = u - G_2 y$

$\Rightarrow y = G_1(u - G_2 y)$

$\Rightarrow (1 + G_1 G_2) y = G_1 u$

$\Rightarrow G_{u\to y} = \frac{G_1}{1 + G_1 G_2} u$

System plant: $\dot{x} = Ax + Bu, \quad y = Cx$

$$\Rightarrow Y(s) = \underbrace{C(sI - A)^{-1} B}_{:= P(s)} \, U(s)$$

$$Y(s) = P(s) U(s)$$

Output feedback Controller:

$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x}) + L(y - C\hat{x})$$

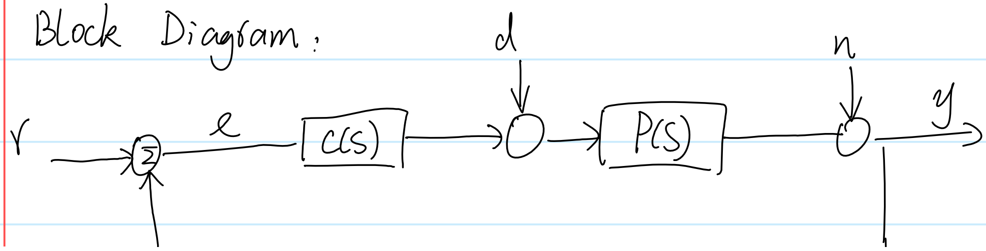
$$u = -K\hat{x}$$

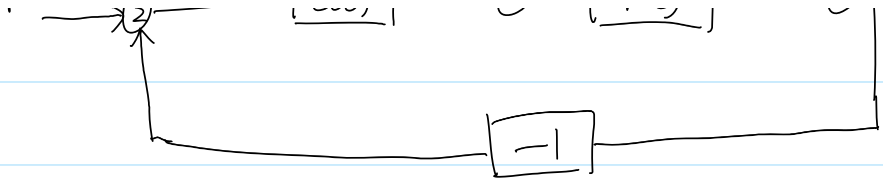
$$\Rightarrow s\hat{X}(s) = (A - BK - LC)\hat{X}(s) + LY(s) + \cancel{\hat{X}(0)} \rightarrow 0$$

$$U(s) = -K\hat{X}(s)$$

$$U(s) = -K\left(sI - (A - BK - LC)\right)^{-1} L \, Y(s) = -C(s) Y(s)$$

Block Diagram:





Closed loop analysis:

$$Y(s) = N(s) + P(s)\big(D(s) + C(s)E(s)\big)$$

$$E(s) = R(s) - Y(s)$$

$$\Rightarrow (1 + P(s)C(s))\,Y(s) = N(s) + P(s)D(s) + P(s)C(s)R(s)$$

$$\Rightarrow Y(s) = \frac{1}{1+PC}N(s) + \frac{P}{1+PC}D(s) + \frac{PC}{1+PC}R(s)$$

$G_{n\to y} = \frac{1}{1+PC}$ (transfer function from n to y)

$G_{d\to y} = \frac{P}{1+PC}$ (〃 〃 〃 d to y)

$G_{r\to y} = \frac{PC}{1+PC}$ (〃 〃 〃 r to y)

Since $E(s) = R(s) - Y(s)$, we can get transfer function $\left.\begin{matrix} n \\ d \\ r \end{matrix}\right\} \to e$ as well.

$$G_{r\to e} = 1 - \frac{PC}{1+PC} = \frac{1}{1+PC}$$