Homework 1

Math 55a, Fall 2018

Due Wednesday, September 12, 2018

Note: the first three problems are warm-up problems on set theory; if you don't know how to do them, ask us!

1. Prove the *pigeon-hole principle*: if A is a finite set, then any injective map $f : A \to A$ is also surjective.

2. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. Give an explicit bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

3. Let F denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$, and let $\mathcal{C} \subset F$ denote the subset of all continuous functions. Prove that $|\mathbb{R}| = |\mathcal{C}| < |F|$.

4. Let $x_1, x_2, \ldots, x_n \in G$ be any elements of a group G. Show that
$$(x_1 x_2 \cdots x_n)^{-1} = x_n^{-1} \cdot x_{n-1}^{-1} \cdots x_2^{-1} \cdot x_1^{-1}.$$

5. Show that a group G cannot be the union of two proper subgroups.

6. Show that any finite group G of even order contains an element $x \in G$ such that $a \neq e$ but $a^2 = e$.

7. Let D_8 be the group of symmetries of a square (including reflections). How many subgroups (including the trivial subgroups D_8 and $\{e\}$) does D_8 have?

8. Let G be a group, and consider the set map $\phi : G \to G$ sending each element $a \in G$ to its square $a^2 \in G$. Show that ϕ is a homomorphism if and only if G is abelian.

9. Let $H \subset G$ be any subgroup of a finite group G. Show that if $|G|/|H| = 2$ then H is a normal subgroup of G.

10. What is the order of the group $GL_2(\mathbb{Z}/3)$ of 2×2 matrices with entries in $\mathbb{Z}/3$ and nonzero determinant?

11. Let G be a group.
    (a) Show that the set of automorphisms of G is itself a group (with group law given by composition). This group is denoted $Aut(G)$.
    (b) For each element $a \in G$, define a map $c_a : G \to G$ by $c_a(x) = axa^{-1}$. Show that c_a is an automorphism of G.
    (c) Show that the map $\phi : G \to Aut(G)$ defined by sending $a \in G$ to $c_a \in Aut(G)$ is a homomorphism.
    (d) Give an example of a group G such that ϕ is an isomorphism.

Supplementary problems: Artin, Chapter 2, problems M6 and M7