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I-7: Feynman rules

1 Introduction

In the previous lecture we saw that scattering calculations are naturally expressed in terms of time-ordered products of fields. The S-matrix has the form

$$\langle f|S|i\rangle \sim \langle \Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle \tag{1}$$

where $|\Omega\rangle$ is the ground state/vacuum in the interacting theory. In this expression the fields $\phi(x)$ are not free but are the full interacting quantum fields. We also saw that in the free theory time-ordered product of two fields is given by the Feynman propagator:

$$D_F(x,y)\equiv\langle 0|T\{\phi_0(x)\,\phi_0(y)\}|\,0\rangle=\lim_{\varepsilon\to 0}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{ik(x-y)} \tag{2}$$

where $|0\rangle$ is the ground state in the free theory.

In this lecture we will develop a method of calculating time-ordered products in the interacting theories in perturbation theory in terms of integrals over various Feynman propagators. There is a beautiful pictorial representation of the perturbation expansion using Feynman diagrams and an associated set of Feynman rules. There are position space Feynman rules, for calculating time-ordered products and also momentum space Feynman rules for calculating S-matrix elements. The momentum space Feynman rules are by far more important – they provide an extremely efficient way to set up calculations of physical results in quantum field theory. They are the main result of this entire first part of the course.

We will first derive the Feynman rules using a Lagrangian formulation of time-evolution and quantization. This is the quickest way to connect Feynman diagrams to classical field theory. We will then derive the Feynman rules again using time-dependent perturbation theory, based on an expansion of the full interacting Hamiltonian around the free Hamiltonian. This calculation much more closely parallels the way perturbation theory is done in quantum mechanics. While the Hamiltonian-based calculation is significantly more involved, it has the distinct advantage that it connects time-evolution directly to a Hermitian Hamiltonian, so time-evolution is guaranteed to be unitary. The Feynman rules resulting from both approaches agree, confirming that they are equivalent (at least in the case of the theory of a real scalar field which is all we have so seen so far). As we progress in our understanding of field theory and encounter particles of different spin and more complicated interactions, unitarity and the requirement of a Hermitian Hamiltonian will play a more important role (see in particular Lecture III-10). A third independent way to derive the Feynman rules is through the path integral (Lecture II-7).

2 Lagrangian derivation

In Lecture I-2 we showed that for free quantum fields,

$$[\phi(\vec{x}\,,t),\phi(\vec{x}\,',t)]=0, \tag{3}$$

$$[\phi(\vec{x}\,,t),\partial_t\phi(\vec{x}\,',t)]=i\hbar\,\delta^3(\vec{x}-\vec{x}\,') \tag{4}$$

and that $(\Box+m^2)\,\phi=0$, which is the Euler-Lagrangian equation for a free scalar field (we have temporarily reinstated $\hbar$ to clarify the classical limit). In an arbitrary interacting theory, we must generalize these equations to specify how the dynamics is determined. In quantum mechanics, this is done with the Hamiltonian. So one natural approach is to assume that $i\partial_t\phi(x)=[\phi,\,H]$ for a interacting quantum field theory, which leads to the Hamiltonian derivation of the Feynman rules in the next section. In this section we discuss the simpler Lagrangian approach based on the Schwinger-Dyson equations, which has the advantage of being manifestly Lorentz invariant from start to finish.

In the Lagrangian approach, Hamilton's equations are replaced by the Euler-Lagrange equations. We therefore assume that our interacting fields satisfy the Euler-Lagrange equations derived from a Lagrangian $\mathcal{L}$ (the generalization of $(\Box + m^2)\,\phi = 0$), just like classical fields. We will also assume Eqs. (3) and (4) are still satisfied. This is a natural assumption, since at any given time the Hilbert space for the interacting theory is the same as that of a free theory. Eq. (3) is a requirement of causality: at the same time but at different points in space all operators, in particular fields, should be simultaneously observable and commute (otherwise there could be faster-than-light communication). This causality requirement will be discussed more in the context of the spin-statistics theorem in Lecture II-5. Eq. (4) is the equivalent of the canonical commutation relation from quantum mechanics: $[\hat{x}\,,\,\hat{p}] = i\,\hbar$. It indicates that a quantity and its time-derivative are not simultaneously observable, the hallmark of the uncertainty principle.

We are now ready to calculate time-ordered products in the interacting theory. At this point we only know how to calculate $\langle 0|T\{\phi(x)\,\phi(x')\}|\,0\rangle$ in the free theory. To calculate this commutator in an interacting theory, it is helpful to have the intermediate result

$$(\Box + m^2)\langle\Omega|T\{\phi(x)\,\phi(x')\}|\Omega\rangle = \langle\Omega|T\{(\Box + m^2)\,\phi(x)\,\phi(x')\}|\Omega\rangle - i\,\hbar\,\delta^4(x - x') \tag{5}$$

where, again, $|\Omega\rangle$ is the vacuum in the interacting theory which may differ from the vacuum $|0\rangle$ in the free theory. This $\delta^4(x - x')$ on the right-side of this equation is critically important: it signifies the difference between the classical and quantum theories, in a way that will be clear shortly.

To derive Eq. (5) we just calculate

$$\partial_t\langle\Omega|T\{\phi(x)\,\phi(x')\}|\Omega\rangle = \partial_t[\langle\Omega|\phi(x)\,\phi(x')|\Omega\rangle\theta(t - t') + \langle\Omega|\phi(x')\,\phi(x)|\Omega\rangle\theta(t' - t)] \tag{6}$$

$$= \langle\Omega|T\{\partial_t\phi(x)\,\phi(x')\}|\Omega\rangle + \langle\Omega|\phi(x)\,\phi(x')|\Omega\rangle\,\partial_t\theta(t - t') + \langle\Omega|\phi(x')\,\phi(x)|\Omega\rangle\,\partial_t\theta(t' - t) \tag{7}$$

$$= \langle\Omega|T\{\partial_t\phi(x)\,\phi(x')\}|\Omega\rangle + \delta(t - t')\langle\Omega|[\phi(x), \phi(x')]|\Omega\,\rangle \tag{8}$$

where we have used $\partial_x\theta(x) = \delta(x)$ in the last line. The second term on the last line vanishes, since $\delta(t - t')$ forces $t = t'$ and $[\phi(x), \phi(x')] = 0$ at equal times. Taking a second time derivative then gives

$$\partial_t^2\langle\Omega|T\{\phi(x)\,\phi(y)\}|\Omega\rangle = \langle\Omega|T\{\partial_t^2\phi(x)\,\phi(x')\}|\Omega\rangle + \delta(t - t')\,\langle\Omega|[\partial_t\phi(x), \phi(x')]|\Omega\rangle$$

Here again $\delta(t - t')$ forces the time to be equal, in which case $[\partial_t\phi(x), \phi(x')] = -i\,\hbar\,\delta^3(\vec{x} - \vec{x}\,')$ as in Eq. (4). Thus,

$$\partial_t^2\langle\Omega|T\{\phi(x)\,\phi(y)\}|\Omega\rangle = \langle\Omega|T\{\partial_t^2\phi(x)\,\phi(x')\}|\Omega\rangle - i\,\hbar\,\delta^4(x - x')$$

and Eq. (5) follows.

For example, in the free theory, $(\Box + m^2)\,\phi_0(x) = 0$. Then Eq. (5) implies

$$(\Box_x + m^2)\,D_F(x, y) = -i\,\hbar\,\delta^4(x - y) \tag{9}$$

which is easy to verify from Eq. (2).

Introducing the notation $\langle\cdots\rangle = \langle\Omega|T\{\cdots\}|\Omega\rangle$ for time-ordered correlation functions in the interacting theory, Eq. (5) can be written as

$$(\Box + m^2)\langle\phi(x)\,\phi(x')\rangle = \langle(\Box + m^2)\,\phi(x)\,\phi(x')\rangle - i\,\hbar\,\delta^4(x - x') \tag{10}$$

It is not hard to see that similar equations hold for commutators involving more fields. We will get $[\partial_t\phi(x), \phi(x_j)]$ terms from the time derivatives acting on the time-ordering operator giving δ-functions. The result is that

$$\Box_x\langle\phi(x)\,\phi(x_1)\cdots\phi(x_n)\rangle = \langle\Box_x\phi(x)\,\phi(x_1)\cdots\phi(x_n)\rangle \tag{11}$$

$$- i\,\hbar\sum_j\,\delta^4(x - x_j)\langle\phi(x_1)\cdots\phi(x_{j\,-1})\,\phi(x_{j+1})\cdots\phi(x_n)\rangle \tag{12}$$

You should check this generalization by calculating $\Box_x\langle\phi(x)\,\phi(x_1)\,\phi(x_2)\rangle$ on your own. Now we use the fact that the quantum field satisfies the same equations of motion as the classical field, by assumption. In particular, if the Lagrangian has the form $\mathcal{L}=-\frac{1}{2}\,\phi(\Box+m^2)\,\phi+\mathcal{L}_{\rm int}[\phi]$ then the (quantum) field satisfies $(\Box+m^2)\,\phi-\mathcal{L}'_{\rm int}[\phi]=0$, where $\mathcal{L}'_{\rm int}[\phi]=\frac{d}{d\phi}\,\mathcal{L}_{\rm int}[\phi]$, giving

$$(\Box_x+m^2)\langle\phi_x\,\phi_1\cdots\phi_n\rangle=\langle\mathcal{L}'_{\rm int}[\phi_x]\,\phi_1\cdots\phi_n\rangle-i\hbar\sum_j\delta^4(x-x_j)\langle\phi_1\cdots\phi_{j-1}\,\phi_{j+1}\cdots\phi_n\rangle \tag{13}$$

where $\phi_x\equiv\phi(x)$ and $\phi_j\equiv\phi(x_j)$. These are known as **Schwinger-Dyson equations**.

The Schwinger-Dyson equations encode the difference between the classical and quantum theories. Note that their derivation did not require any specification of the dynamics of the theory, only that the canonical commutation relations in Eq. (4) are satisfied. In particular, in a classical theory, $[\phi(\vec{x}\,',t),\partial_t\phi(\vec{x}\,,t)]=0$ and therefore classical time-ordered correlation functions would satisfy a similar equation but without the $\delta^4(x-x_j)$ terms (i.e. $\hbar=0$). That is, in a classical theory, correlation functions satisfy the same differential equations as the fields within the correlation functions. In a quantum theory, that is true only up to δ-functions, which in this context are also called **contact interactions**. These contact interactions allow virtual particles to be created and destroyed, which permits closed loops to form in the Feynman diagrammatic expansion, as we will now see.

2.1 Position space Feynman rules

The Schwinger-Dyson equations specify a completely non-perturbative relationship among correlation functions in the fully interacting theory. Some non-perturbative implications will be discussed in later lectures (in particular II-7). In this section, we will solve the Schwinger-Dyson equations in perturbation theory.

For efficiency, we write $\delta_{xi}=\delta^4(x-x_i)$ and $D_{ij}=D_{ji}=D_F(x_i,x_j)$. We will also set $m=0$ for simplicity (the $m\neq0$ case is a trivial generalization), and $\hbar=1$. With this notation the Green's function equation for the Feynman propagator can be written concisely as

$$\Box_x D_{x1}=-i\,\delta_{x1} \tag{14}$$

This relation can be used to rewrite correlation functions in a suggestive form. For example, the two-point function can be written as

$$\langle\phi_1\,\phi_2\rangle=\int d^4x\,\delta_{x1}\langle\phi_x\,\phi_2\rangle=i\int d^4x(\Box_x D_{x1})\langle\phi_x\,\phi_2\rangle=i\int d^4x\,D_{x1}\Box_x\langle\phi_x\,\phi_2\rangle \tag{15}$$

where we have integrated by parts in the last step. This is suggestive because $\Box_x$ acting on a correlator can be simplified with the Schwinger-Dyson equations.

Now first suppose we are in the free theory where $\mathcal{L}_{\rm int}=0$. Then the 2-point function can be evaluated using the Schwinger-Dyson equation $\Box_x\langle\phi_x\,\phi_y\rangle=-i\,\delta_{xy}$ to give

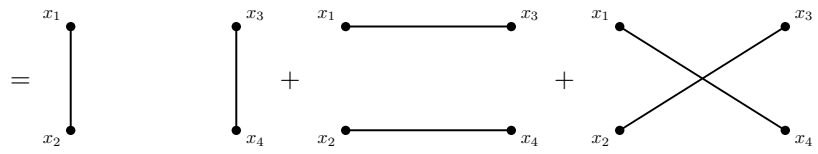
$$\langle\phi_1\,\phi_2\rangle=\int d^4x\,D_{x1}\,\delta_{x2}=D_{12} \tag{16}$$

as expected. For a 4-point function, the expansion is similar

$$\langle\phi_1\ \phi_2\ \phi_3\ \phi_4\rangle=i\int d^4\,x\,D_{x1}\Box_x\langle\phi_x\ \phi_2\ \phi_3\ \phi_4\rangle=\int d^4\,x\,D_{x1}\{\delta_{x2}\langle\phi_3\ \phi_4\rangle+\delta_{x3}\langle\phi_2\ \phi_4\rangle+\delta_{x4}\langle\phi_2\,\phi_3\rangle\} \tag{17}$$

Collapsing the δ-functions and using Eq.(16) this becomes

$$\langle\phi_1\,\phi_2\,\phi_3\,\phi_4\rangle=D_{12}\,D_{34}+D_{13}\,D_{24}+D_{14}\,D_{23} \tag{18}$$



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Each of these terms is drawn as a diagram. In the diagrams the points $x_1 \cdots x_4$ correspond to points where the correlation function is evaluated and the lines connecting these points correspond to propagators.

Next, we'll add interactions. Consider for example the 2-point function again with Lagrangian $\mathcal{L} = -\frac{1}{2}\,\phi\Box\phi + \frac{g}{3!}\,\phi^3$ (the 3! is a convention which will be justified shortly). Up to Eq.(15) things are the same as before. But now an application of the Schwinger-Dyson equations involves $\mathcal{L}'_{\rm int}[\phi] = \frac{g}{2}\,\phi^2$, so we get

$$\langle \phi_1\,\phi_2\rangle = i\int d^4\,x\,D_{1\,x}\Big(\frac{g}{2}\langle\phi_x^2\,\phi_2\rangle - i\,\delta_{x\,2}\Big) \tag{20}$$

To simplify this, we introduce another integral, use $\delta_{2\,y} = i\Box_y D_{y\,2}$, and integrate by parts again to give

$$\langle \phi_1\,\phi_2\rangle = D_{1\,2} - \frac{g}{2}\int d^4\,x\,d^4\,y\,D_{x\,1}\,D_{y\,2}\Box_y\langle\phi_x^2\,\phi_y\rangle \tag{21}$$

$$= D_{1\,2} - \frac{g^2}{4}\int d^4\,x\,d^4\,y D_{x\,1}\,D_{2\,y}\langle\phi_x^2\,\phi_y^2\rangle + i\,g\int d^4\,x\,D_{1\,x}\,D_{2\,x}\langle\phi_x\rangle \tag{22}$$

If we are only interested in order g^2, the $\langle\phi_x^2\,\phi_y^2\rangle$ term can then be simplified using the free field Schwinger-Dyson result, Eq.(18),

$$\langle\phi_x^2\,\phi_y^2\rangle = 2\,D_{x\,y}^2 + D_{x\,x}\,D_{y\,y} + \mathcal{O}(g) \tag{23}$$

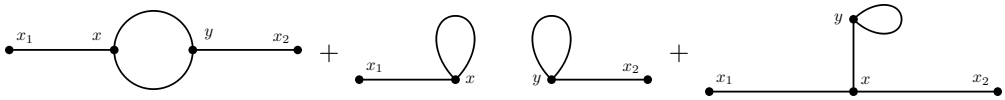
The $\langle\phi_x\rangle$ term in can be expanded using the Schwinger-Dyson equations again

$$\langle\phi_x\rangle = i\int d^4\,y D_{x\,y}\Box_y\langle\phi_y\rangle = i\,\frac{g}{2}\int d^4\,y D_{x\,y}\langle\phi_y^2\rangle = i\,\frac{g}{2}\int d^4\,y D_{x\,y}\,D_{y\,y} + \mathcal{O}(\lambda^2) \tag{24}$$

Thus the final result is

$$\langle\phi_1\;\;\phi_2\rangle \;=\; D_{1\,2} \;-\; g^2\;\int d^4\;\;x\,d^4\;\;y\;\;\left(\frac{1}{2}\;\;D_{1\,x}\;\;D_{x\,y}^2\;\;D_{y\,2}\;+\;\frac{1}{4}\;\;D_{1\,x}\;\;D_{x\,x}\;\;D_{y\,y}\;\;D_{y\,2}\;+\;\frac{1}{2}\,D_{1\,x}\,D_{2\,x}\,D_{x\,y}\,D_{y\,y}\right) \tag{25}$$

The three new terms correspond to the diagrams



(26)

These diagrams now have new points, labeled x and y, which are integrated over.

From these examples, and looking at the pictures, it is easy to infer the way the perturbative expansion will work for higher-order terms or more general interactions.

1. Start with (external) points x_i for each position at which fields in the correlation function are evaluated. Draw a line from each point.
2. A line can then either contract to an existing line, giving a Feynman propagator connecting the endpoints of the two lines, or it can split due to an interaction. A split gives a new (internal) vertex proportional to the coefficient of $\mathcal{L}'_{\rm int}[\phi]$ times i and new lines corresponding to the fields in $\mathcal{L}'_{\rm int}[\phi]$.
3. At a given order in the perturbative couplings, the result is the sum of all diagrams with all the lines contracted, integrated over the positions of internal vertices.

These are known as the **position space Feynman rules**. The result is a set of diagrams. The original time-ordered product is given by a sum over integrals represented by the diagrams with an appropriate numerical factor. To determine the numerical factor, it is conventional to write interactions normalized by the number of permutations of identical fields, for example

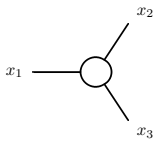
$$\mathcal{L}_{\rm int} \;\;=\;\; \frac{\lambda}{4!}\,\phi^4,\;\;\;\frac{g}{3!}\,\phi^3,\;\;\;\frac{\kappa}{5!3!2!}\,\phi_1^5\,\phi_2^3\,\phi_3^2,\;\;\;\cdots \tag{27}$$

Thus, when the derivative is taken to turn the interaction into a vertex, the prefactor becomes $\frac{1}{(n-1)!}$. This $(n-1)!$ is then canceled by the number of permutations of the lines coming out of the vertex, not including the line coming in which we already fixed. In this way, the $n!$ factors all cancel. The diagram is therefore associated with just the prefactor λ, g, κ, etc. from the interaction.

In some cases, such as theories with real scalar fields, some of the permutations give the same amplitude. For example, if a line connects back to itself, than permuting the two legs gives the same integral. In this case, a factor of $\frac{1}{2}$ in the normalization is not canceled, so we must divide by 2 to get the prefactor for a diagram. That is why the third diagram in Eq. (26) has a $\frac{1}{2}$ and the second diagram has a $\frac{1}{4}$. For the first diagram, the factor of $\frac{1}{2}$ comes from exchanging the two lines connecting x and y. So there is one more rule

4. Drop all the $n!$ factors in the coefficient of the interaction, but then divide by the geometrical symmetry factor for each diagram.

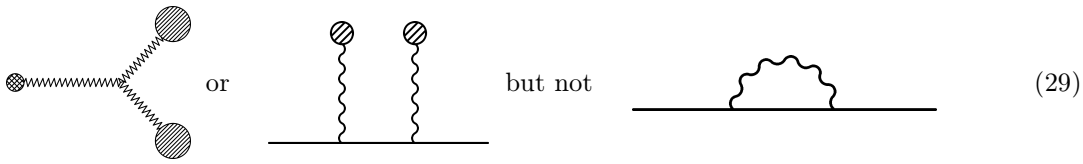
Symmetries are ways that a graph can be deformed so that it looks the same with the *external points, labeled x_i held fixed*. Thus, while there are symmetry factors for the graphs in Eq. (26), a graph like



(28)

has no symmetry factor, since the graph cannot be brought back to itself without tangling up the external lines. The safest way to determine the symmetry factor is simply to write down all the diagrams using the Feynman rules and see which give the same integrals. In practice, diagrams almost never have geometric symmetry factors; occasionally in theories with scalars there are factors of 2.

As mentioned in the introduction, an advantage of this approach is that it provides an intuitive way to connect and contrast the classical and quantum theories. In a classical theory, as noted above, the contact interactions are absent. It was these contact interactions that allowed us to *contract* two fields within a correlation function to produce a term in the expansion with fewer fields. For example, $\Box\langle\phi_1\,\phi_2\,\phi_3\,\phi_4\rangle = i\,\delta_{12}\langle\phi_3\,\phi_4\rangle + \cdots$. In the classical theory, all that can happen is that the fields will proliferate. Thus we can have diagrams like



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The first process may represent general relativistic corrections to Mercury's orbit (Lecture I-3), which can be calculated entirely with classical field theory. The external points in this case are all given by external sources, such as Mercury or the Sun which are illustrated with the blobs. The second process represents an electron in an external electromagnetic field (studied in quantum mechanics in Lecture I-4). This is a semi-classical process in which a single field is quantized (the electron) and does not get classical-source blobs on the end of its lines. But since quantum mechanics is first-quantized, particles cannot be created or destroyed and no closed loops can form. Thus, neither of these first two diagrams involve virtual pair creation. The third describes a process which can only be described with quantum field theory (or, with difficulty, with old-fashioned perturbation theory as discussed in Lecture I-4). It is a Feynman diagram for the electron self energy, which will be calculated properly using quantum field theory in Lecture III-5.

3 Hamiltonian derivation

In this section, we reproduce the position-space Feynman rules using time-dependent perturbation theory. Instead of assuming that the quantum field satisfies the Euler-Lagrange equations, we instead assume its dynamics is determined by a Hamiltonian H by the Heisenberg equations of motion $i\partial_t\phi(x) = [\phi, H]$. The formal solution of this equation is

$$\phi(\vec{x}\,,t) = S(t,t_0)^\dagger\,\phi(\vec{x}\,)S(t,t_0) \tag{30}$$

where $S(t,t_0)$ is the time-evolution operator (the S-matrix) which satisfies

$$i\partial_t S(t,t_0) = H(t)S(t,t_0) \tag{31}$$

These are the dynamical equations in the **Heisenberg picture** where all the time dependence is in operators. States including the vacuum state $|\Omega\rangle$ in the Heisenberg picture are, by definition, time independent. As mentioned in Lecture I-2, the Hamiltonian can either be defined at any given time as a functional of the fields $\phi(\vec{x}\,)$ and $\pi(\vec{x}\,)$ or equivalently as a functional of the creation and annihilation operators $a_p^\dagger$ and a_p. We will not need an explicit form of the Hamiltonian for this derivation so we just assume it is some time-dependent operator $H(t)$.

The first step in time-dependent perturbation theory is to write the Hamiltonian as

$$H(t) = H_0 + V(t) \tag{32}$$

where the time-evolution induced by H_0 can be solved exactly and V is small in some sense. For example, H_0 could be the free Hamiltonian, which is time-independent, and V might be a ϕ^3 interaction

$$V(t) = \int d^3x\,\frac{g}{3!}\,\phi(\vec{x}\,,t)^3 \tag{33}$$

The operators $\phi(\vec{x}\,,t)$, H, H_0 and V are all in the Heisenberg picture.

To do time-dependent perturbation theory we will change to the **interaction picture**. In the interaction picture the fields evolve only with H_0. The interaction picture fields are just what we had been calling (and will continue to call) the free fields

$$\phi_0(\vec{x}\,,t) = e^{iH_0(t-t_0)}\,\phi(\vec{x}\,)\,e^{-iH_0(t-t_0)} = \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,\omega_p}}\big(a_p\,e^{-ipx} + a_p^\dagger\,e^{ipx}\big) \tag{34}$$

To be precise, $\phi(\vec{x}\,)$ is the Schrödinger picture field, which does not change with time. The free fields are equal to the Schrödinger picture fields and also to the Heisenberg picture fields, by definition, at a single reference time which we call t_0.

Using Eq. (30), we see that the Heisenberg picture fields are related to the free fields by

$$\phi(\vec{x}\,,t) = S^\dagger(t,t_0)\,e^{-iH_0(t-t_0)}\,\phi_0(\vec{x}\,,t)\,e^{iH_0(t-t_0)}\,S(t,t_0) \tag{35}$$

$$= U^\dagger(t,t_0)\,\phi_0(\vec{x}\,,t)\,U(t,t_0) \tag{36}$$

The operator $U(t,\,t_0) \equiv e^{iH_0(t-t_0)}\,S(t,\,t_0)$ therefore relates the full Heisenberg-picture fields to the free fields at the *same time* t. The evolution begins from the time t_0 where the fields in the two pictures (and the Schrödinger picture) are equal.

We can find a differential equation for $U(t,t_0)$ using Eq.(31):

$$\begin{aligned} i\partial_t U(t,t_0) &= i\big(\partial_t e^{iH_0(t-t_0)}\big)S(t,t_0) + e^{iH_0(t-t_0)}\,i\partial_t S(t,t_0) \\ &= -e^{iH_0(t-t_0)}\,H_0\,S(t,t_0) + e^{iH_0(t-t_0)}\,H\,(t)S(t,t_0) \\ &= e^{iH_0(t-t_0)}[-H_0 + H(t)]\,e^{-iH_0(t-t_0)}\,e^{iH_0(t-t_0)}\,S(t,t_0) \\ &= V_I(t)\,U(t,t_0) \end{aligned} \tag{37}$$

where $V_I(t) \equiv e^{iH_0(t-t_0)}\,V(t)\,e^{-iH_0(t-t_0)}$ is the original Heisenberg picture potential $V(t)$ from Eq. (32), now expressed in the interaction picture.

If everything commuted, the solution to Eq. (37) would be $U(t,t_0) = \exp\left(-i\int_{t_0}^{t} V_I(t')dt'\right)$. But $V_I(t_1)$ does not necessarily commute with $V_I(t_2)$, so this is not the right answer. It turns out the right answer is very similar

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} dt'\, V_I(t')\right]\right\} \tag{38}$$

where $T\{\}$ is the time-ordering operator, introduced in the last lecture. This solution works because time-ordering effectively makes everything inside commute:

$$T\{A\cdots B\cdots\} = T\{B\cdots A\cdots\} \tag{39}$$

Taking the derivative, you can see immediately that Eq. (38) satisfies Eq. (37). Since it has the right boundary conditions, namely $U(t,t)=1$, this solution is unique.

Time ordering of an exponential is defined in the obvious way through its expansion

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\, V_I(t') - \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T\{V_I(t')\, V_I(t'')\} + \cdots \tag{40}$$

This is known as a **Dyson series**. Dyson defined the time-ordered product and this series in his classic paper from 1949. In that paper he showed the equivalence of old-fashioned perturbation theory, or more exactly, the interaction picture method developed by Schwinger and Tomonaga based on time-dependent perturbation theory, and Feynman's method, involving space-time diagrams, which we are about to get to.

3.1 Perturbative solution for the Dyson series

We guessed and checked the solution to Eq. (37), which is often the easiest way to solve a differential equation. It is perhaps illuminating also to solve it directly, using perturbation theory.

Removing the subscript on V for simplicity, the differential equation we want to solve is

$$i\partial_t U(t,t_0) = V(t)\, U(t,t_0) \tag{41}$$

Integrating this equation lets us write it in an equivalent form

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\, V(t')\, U(t',t_0) \tag{42}$$

where 1 is the appropriate integration constant so that $U(t_0,t_0)=1$.

Now we will solve the integral equation order-by-order in V. At zeroth order in V,

$$U(t,t_0) = 1 \tag{43}$$

To first order in V we find

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\, V(t') + \cdots \tag{44}$$

To second order,

$$\begin{aligned} U(t,t_0) \;&= 1 - i\int_{t_0}^{t} dt'\, V(t')\left[1 - i\int_{t_0}^{t'} dt''\, V(t'') + \cdots\right] \\ &= 1 - i\int_{t_0}^{t} dt'\, V(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, V(t')V(t'') + \cdots \end{aligned} \tag{45}$$

The second integral has $t_0 < t'' < t' < t$, which is the same as $t_0 < t'' < t$ and $t'' < t' < t$. So it can also be written as

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, V(t')V(t'') = \int_{t_0}^{t} dt'' \int_{t''}^{t} dt'\, V(t')V(t'') = \int_{t'}^{t} dt'' \int_{t_0}^{t} dt'\, V(t'')V(t') \tag{46}$$

where we have relabeled $t'' \leftrightarrow t'$ and swapped the order of the integrals to get the third form. Averaging the first and third form gives

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V(t')V(t'') = \frac{1}{2}\int_{t_0}^{t} dt' \left[\int_{t_0}^{t'} dt'' V(t')V(t'') + \int_{t'}^{t} dt'' V(t'')V(t') \right] \tag{47}$$

$$= \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T\{V(t')\,V(t'')\} \tag{48}$$

Thus,

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\, V(t') + \frac{(-i)^2}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T\{V(t')V(t'')\} + \cdots \tag{49}$$

Continuing this way, we find, restoring the subscript on V, that

$$U(t,t_0) = T\left\{ \exp\left[-i\int_{t_0}^{t} dt'\, V_I(t') \right] \right\} \tag{50}$$

3.2 U relations

It is convenient to abbreviate U with

$$U_{21} \equiv U(t_2,t_1) = T\left\{ \exp\left[-i\int_{t_1}^{t_2} dt'\, V_I(t') \right] \right\} \tag{51}$$

Remember that in field theory we always have later times on the left. It follows that

$$U_{21}\,U_{12} = 1 \quad \Rightarrow \quad U_{21}^{-1} = U_{21}^{\dagger} = U_{12} \tag{52}$$

and for $t_1 < t_2 < t_3$

$$U_{32}\,U_{21} = U_{31} \tag{53}$$

Multiplying this by U_{12} on the right, we find

$$U_{31}\,U_{12} = U_{32} \tag{54}$$

which is the same identity with $2 \leftrightarrow 1$. Multiplying Eq. (53) by U_{23} on the left, gives the same identity with $3 \leftrightarrow 1$. Therefore, this identity holds for any time-ordering.

So our defining relation, Eq.(36):

$$\phi(\vec{x}\,,t) = U^{\dagger}(t,t_0)\;\phi_0(\vec{x}\,,t)\,U(t,t_0) \tag{55}$$

lets us write

$$\phi(x_1) = \phi(\vec{x}_1,t_1) = U_{10}^{\dagger}\;\phi_0(\vec{x}_1,t_1)\,U_{10} = U_{01}\;\phi_0(x_1)\,U_{10} \tag{56}$$

3.3 Vacuum matrix elements

The next thing to discuss is the vacuum state $|\Omega\rangle$. In deriving LSZ we used that this state was annihilated by the operators $a_p(t)$ in the interacting theory at a time $t = -\infty$. To relate this to a state for which we know how the free field creation and annihilation operators act, we need to evolve it to the reference time t_0 where the free and interacting pictures are taken equal. This is straightforward: states evolve (in the Schrödinger picture) with $S(t,\,t_0)$, and thus $S(t,\,t_0)|\Omega\rangle$ is annihilated by $a_p(t_0)$ at $t = -\infty$. Equivalently (in the Heisenberg picture) the operator $a_p(t) = S(t,t_0)^{\dagger}\,a_p(t_0)S(t,t_0)$ annihilates $|\Omega\rangle$ at $t = -\infty$.

In the free theory, there is a state $|0\rangle$ which is annihilated by the a_p. Since the a_p evolve with a simple phase rotation, the same state $|0\rangle$ is annihilated by the (free theory) a_p at any time. More precisely, if we do not assume $|0\rangle$ has zero energy, then $a_p(t_0)\,e^{iH_0(t-t_0)}|0\rangle = 0$ at $t = -\infty$. Since at the time t_0 the free and interacting theory creation and annihilation operators are equal, the a_p in both theories annihilate $e^{iH_0(t-t_0)}|0\rangle$ and $S(t,\,t_0)|\Omega\rangle$. Thus the two states must be proportional. Therefore

$$|\Omega\rangle = \mathcal{N}_i \lim_{t\to-\infty} S^{\dagger}(t,t_0)\,e^{iH_0(t-t_0)}|0\rangle = \mathcal{N}_i U_{0-\infty}|0\rangle \tag{57}$$

for some number $\mathcal{N}_i$. Similarly, $\langle\Omega| = \mathcal{N}_f\langle 0|\, U_{\infty 0}$ for some number $\mathcal{N}_f$.

Now let's see what happens when we rewrite correlation functions in the interaction picture. We are interested in time-ordered products $\langle\Omega|T\{\phi(x_1)\ \cdots\ \phi(x_n)\}|\Omega\rangle$. Since all the $\phi(x_i)$ are within a time-ordered product, we can write them in any order we want. So let us put them in time order, or equivalently, we assume $t_1 > \cdots > t_n$ without loss of generality. Then

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \langle\Omega|\phi(x_1)\cdots\phi(x_n)|\Omega\rangle \tag{58}$$

$$=$$

$$\mathcal{N}_i\mathcal{N}_f\langle 0|U_{\infty 0}\,U_{01}\,\phi_0(x_1)\,U_{10}\,U_{02}\,\phi_0(x_2)\,U_{20}\cdots U_{0n}\,\phi_0(x_n)\,U_{n0}\,U_{0-\infty}|\,0\rangle \tag{59}$$

$$= \mathcal{N}_i\mathcal{N}_f\langle 0|U_{\infty 1}\,\phi_0(x_1)\,U_{12}\,\phi_0(x_2)\,U_{23}\cdots U_{(n-1)n}\,\phi_0(x_n)\,U_{n-\infty}|\,0\rangle \tag{60}$$

Now, since the t_i are in time order and the U_{ij} are themselves time-ordered products involving times between t_i and t_j, everything in this expression is in time order. Thus

$$\begin{aligned}\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle &= \mathcal{N}_i\mathcal{N}_f\langle 0|T\{U_{\infty 1}\,\phi_0(x_1)\,U_{12}\,\phi_0(x_2)\,U_{23}\cdots\phi_0(x_n)\,U_{n-\infty}\}|\,0\rangle \\ &= \mathcal{N}_i\mathcal{N}_f\langle 0|T\{\phi_0(x_1)\cdots\phi_0(x_n)\,U_{\infty,-\infty}\}|\,0\rangle\end{aligned} \tag{61}$$

The normalization should set so that $\langle\Omega|\Omega\rangle = 1$, just like $\langle 0|\ 0\rangle = 1$ in the free theory. This implies $\mathcal{N}_i\mathcal{N}_f = \langle 0|\,U_{\infty-\infty}|\,0\rangle^{-1}$ and therefore

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_0(x_1)\cdots\phi_0(x_n)\,U_{\infty,-\infty}\}|\,0\rangle}{\langle 0|\,U_{\infty,-\infty}|0\rangle} \tag{62}$$

Substituting in Eq. (51) we then get

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\big\langle 0\big|T\big\{\phi_0(x_1)\cdots\phi_0(x_n)\exp[-i\textstyle\int_{-\infty}^{\infty}dt\,V_I(t)\,]\big\}\big|0\big\rangle}{\big\langle 0\big|T\big\{\exp\,[-i\textstyle\int_{-\infty}^{\infty}dt\,V_I(t)\,]\big\}\big|0\big\rangle} \tag{63}$$

3.4 Interaction potential

The only thing left to understand is what $V_I(t)$ is. We have defined the time t_0 as when the interacting fields are the same as the free fields. For example, a cubic interaction would be

$$V(t_0) = \int d^3x\,\frac{g}{3!}\,\phi(\vec{x}\,,t_0)^3 = \int d^3x\,\frac{g}{3!}\,\phi_0(\vec{x}\,,t_0)^3 = \int d^3x\,\frac{g}{3!}\,\phi(\vec{x}\,)^3 \tag{64}$$

Recall that the time dependence of the free fields is determined by the free Hamiltonian

$$\phi_0(\vec{x}\,,t) = e^{iH_0(t-t_0)}\,\phi_0(\vec{x}\,)\,e^{-iH_0(t-t_0)} \tag{65}$$

and therefore

$$V_I = e^{iH_0(t-t_0)}\left[\int d^3x\,\frac{g}{3!}\,\phi_0(\vec{x}\,)^3\right]e^{-iH_0(t-t_0)} = \int d^3x\,\frac{g}{3!}\,\phi_0(\vec{x}\,,t)^3 \tag{66}$$

So the interaction picture potential is expressed in terms of the free fields at all times.

Now we will make our final transition away from non-Lorentz invariant Hamiltonians to Lorentz-invariant Lagrangians, leaving old-fashioned perturbation theory for good. Recall that the potential is related to the Lagrangian by $V_I = -\int\ d^3x\mathcal{L}_{\rm int}[\phi_0]$ where $\mathcal{L}_{\rm int}$ is the interacting part of the Lagrangian density. Then,

$$U_{\infty,-\infty} = \exp\left[-i\int_{-\infty}^{\infty}dt\,V_I(t)\right] = \exp\left[i\int_{-\infty}^{\infty}d^4x\,\mathcal{L}_{\rm int}[\phi_0]\right] \tag{67}$$

The $\int_{-\infty}^{\infty}dt$ combined with the $\int\ d^3x$ to give a Lorentz invariant integral.

In summary, matrix elements of interacting fields in the interacting vacuum are given by

$$\langle\Omega|\phi(x_1)\cdots\phi(x_n)|\Omega\rangle = \frac{\langle 0|U_{\infty 1}\,\phi_0(x_1)\,U_{12}\,\phi_0(x_2)\,U_{23}\cdots\phi_0(x_n)\,U_{n,-\infty}|\,0\rangle}{\langle 0|U_{\infty,-\infty}|\,0\rangle} \tag{68}$$

where $|\Omega\rangle$ is the ground state in the interacting theory and

$$U_{ij} = T\left\{\exp\left[i\int_{t_j}^{t_i} d^4x\, \mathcal{L}_{\rm int}[\phi_0]\right]\right\} \tag{69}$$

with $\mathcal{L}_{\rm int}[\phi] = \mathcal{L}[\phi] - \mathcal{L}_0[\phi]$, where $\mathcal{L}_0[\phi]$ is the free Lagrangian. The free Lagrangian is defined as whatever goes into the free field evolution, usually taken to be just kinetic terms.

For the special case of time-ordered products, such as what we need for S-matrix elements, this simplifies to

$$\boxed{\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\left\langle 0\middle|T\left\{\phi_0(x_1)\cdots\phi_0(x_n)\, e^{i\int d^4x\,\mathcal{L}_{\rm int}[\phi_0]}\right\}\middle|0\right\rangle}{\left\langle 0\middle|T\left\{e^{i\int d^4x\,\mathcal{L}_{\rm int}[\phi_0]}\right\}\middle|0\right\rangle}} \tag{70}$$

Which is a remarkably simple and manifestly Lorentz invariant result.

3.5 Time-ordered products and contractions

We will now see that the expansion of Eq. (70) produces the same position space Feynman rules as those coming from the Lagrangian approach described in Section 2. To see that, let's take as an example our favorite ϕ^3 theory with interaction Lagrangian

$$\mathcal{L}_{\rm int}[\phi] = \frac{g}{3!}\,\phi^3 \tag{71}$$

and consider $\langle\Omega|T\{\phi(x_1)\,\phi(x_2)\}|\Omega\rangle$.

The numerator of Eq.(70) can be expanded perturbatively in λ as

$$\left\langle 0\middle|T\left\{\phi_0(x_1)\,\phi_0(x_2)\, e^{i\int d^4x\,\mathcal{L}_{\rm int}[\phi_0]}\right\}\middle|0\right\rangle = \left\langle 0\middle|T\left\{\phi_0(x_1)\,\phi_0(x_2)\right\}\middle|0\right\rangle \tag{72}$$

$$+\frac{ig}{3!}\int d^4x \left\langle 0\middle|T\left\{\phi_0(x_1)\,\phi_0(x_2)\,\phi_0(x)^3\right\}\middle|0\right\rangle \tag{73}$$

$$+\left(\frac{ig}{3!}\right)^2\frac{1}{2}\int d^4x\int d^4y\left\langle 0\middle|T\left\{\phi_0(x_1)\,\phi_0(x_2)\,\phi_0(x)^3\,\phi_0(y)^3\right\}\middle|0\right\rangle + \cdots \tag{74}$$

A similar expansion would result from any time-ordered product of interacting fields. Thus we now only need to evaluate correlation functions of products of free fields.

To evaluate correlation functions of free fields, is it helpful to write $\phi_0(x) = \phi_+(x) + \phi_-(x)$ where

$$\phi_+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\,a_p^\dagger\, e^{ipx}, \quad \phi_-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\,a_p\, e^{-ipx} \tag{75}$$

with ϕ_+ containing only creation operators and ϕ_- only annihilation operators. Then products of ϕ_0 fields at different points become sums of products of ϕ_+ and ϕ_- fields at different points. For example,

$$\begin{aligned}&\left\langle 0\middle|T\left\{\phi_0(x_1)\,\phi_0(x_2)\,\phi_0(x)^3\,\phi_0(y)^3\right\}\middle|0\right\rangle\\ =&\langle 0\middle|T\left\{[\phi_+(x_1)+\phi_-(x_1)][\phi_+(x_2)+\phi_-(x_2)][\phi_+(x)+\phi_-(x)]^3[\phi_+(y)+\phi_-(y)]^3\right\}\middle|0\rangle \qquad (76)\\ =&\langle 0\middle|T\left\{\phi_+(x_1)\,\phi_+(x_2)\,\phi_+(x)^3\,\phi_+(y)^3\right\}\middle|0\rangle + 2\langle 0\middle|T\left\{\phi_+(x_2)\,\phi_+(x_1)\,\phi_+(x)^3\,\phi_+(y)^2\,\phi_-(y)\right\}\middle|0\rangle + \cdots\end{aligned}$$

The last line is supposed to indicate that the result is the sum of a set of products of ϕ_+ and ϕ_- operators evaluated at different points. In each element of this sum, a ϕ_+ would create a particle which, to give a non-zero result, must then be annihilated by some ϕ_- operator. The matrix element can only be non-zero if every particle that is created is destroyed, so every term must have 4 ϕ_+ operators and 4 ϕ_- operators. Each pairing of ϕ_+ with ϕ_- to get a Feynman propagator is called a **contraction** (not to be confused with a Lorentz contraction). The result is then the sum of all possible contractions.

Each contraction represents the creation and then annihilation of a particle, with the creation happening at an earlier time then the annihilation. Each contraction gives a factor of the Feynman propagator

$$\langle 0|T\{\phi_0(x)\,\phi_0(y)\}|\,0\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\,e^{i\,k\,(x-y)} \equiv D_F(x,y) \tag{77}$$

A time-ordered correlation function of free fields is given given by a sum over all possible ways in which all of the fields in the product can be contracted with each other. This is a result known as **Wick's theorem**, the proof of which is given in an appendix to this lecture.

To see how Wick's theorem works, let us return to our example and use the notation $D_{ij} \equiv D_F(x_i, x_j)$. The first term in the expansion of $\langle\Omega|T\{\phi(x_1)\,\phi(x_2)\}|\Omega\rangle$ is $\langle 0|T\{\phi_0(x_1)\,\phi_0(x_2)\}|\,0\rangle$, from Eq.(72). There is only one contraction here, which gives the propagator $D_F(x_1,x_2) = D_{12}$. The second term, in Eq. (73) has an odd number of ϕ fields, and therefore cannot be completely contracted and must vanish. The third term, in Eq.(74) involves 6 fields, and there are multiple possible contractions.

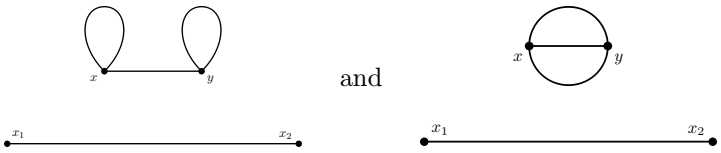
$$\begin{aligned}
&\langle 0|T\{\phi_0(x_1)\,\phi_0(x_2)\,\phi_0(x)\,\phi_0(x)\,\phi_0(x)\,\phi_0(y)\,\phi_0(y)\,\phi_0(y)\}|0\rangle \\
&\quad = 9\,D_{12}\,D_{xx}\,D_{xy}\,D_{yy} + 6\,D_{12}\,D_{xy}^3 \\
&\qquad + 18\,D_{1x}\,D_{2x}\,D_{xy}\,D_{yy} + 9\,D_{1x}\,D_{2y}\,D_{xx}\,D_{yy} + 18\,D_{1x}\,D_{2y}\,D_{xy}^2 \\
&\qquad\quad + 18\,D_{1y}\,D_{2y}\,D_{xy}\,D_{xx} + 9\,D_{1y}\,D_{2x}\,D_{xx}\,D_{yy} + 18\,D_{1y}\,D_{2x}\,D_{xy}^2
\end{aligned} \tag{78}$$

As in Eq. (74), we have to integrate over x and y, thus many of these terms (those on the last line) give the same contributions as other terms. Thus we find, at next-to-leading order in λ,

$$\Big\langle\Omega\Big|\,T\{\phi(x_1)\,\phi(x_2)\}\Big|\Omega\Big\rangle = \frac{1}{\Big\langle 0\Big|T\{e^{i\int\mathcal{L}_{\rm int}}\}\Big|0\Big\rangle}\Big\{D_{12} - g^2\int d^4x\int d^4y \tag{79}$$

$$\times\Big[\,\frac{1}{8}\;D_{12}\;D_{xx}\;D_{xy}\;D_{yy} + \frac{1}{12}\;D_{12}\;D_{xy}^3 + \frac{1}{2}\;D_{1x}\;D_{2x}\;D_{xy}\;D_{yy} + \frac{1}{4}\;D_{1x}\;D_{xx}\;D_{yy}\;D_{y2} + \frac{1}{2}\,D_{1x}\,D_{xy}^2\,D_{y2}\Big]\Big\}$$

The position space Feynman rules which connect this expansion to diagrams are the same as those coming from the Lagrangian approach in Section 2. Comparing to Eq. (25) we see that the sum of terms is exactly the same, including combinatoric factors, except for two differences: the $\langle 0|T\{e^{i\int\mathcal{L}_{\rm int}}\}|\,0\rangle$ factor and the first two terms on the second line which correspond to diagrams



(80)

These two differences precisely cancel.

To see the cancellation, note that the extra diagrams both include **bubbles**, that is, they have connected subgraphs not involving any external point. The bubbles are exactly what is summed by the the denominator of Eq. (70). To see this, note that Wick's theorem also applies to the denominator of Eq. (70). Up to order λ^2, it gives

$$\Big\langle 0\Big|T\Big\{e^{i\int d^4x\,\mathcal{L}_{\rm int}[\phi_0]}\Big\}\Big|0\Big\rangle = \langle 0|\,0\rangle + \left(\frac{\rm ig}{3!}\right)^2\frac{1}{2}\int d^4x\int d^4y\Big\langle 0\Big|T\Big\{\,\phi_0(x)^3\,\phi_0(y)^3\,\Big\}\Big|0\Big\rangle + \cdots \tag{81}$$

We have dropped the $\mathcal{O}(\lambda)$ term since it involves an odd number of fields and therefore vanishes by Wick's theorem. Performing a similar expansion as above, we find

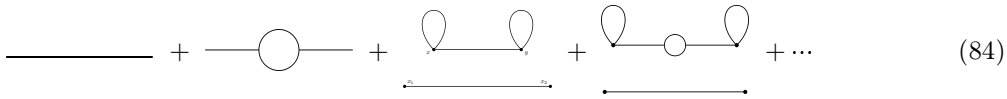
$$\Big\langle 0\Big|T\Big\{e^{i\int d^4x\,\mathcal{L}_{\rm int}[\phi_0]}\Big\}\Big|0\Big\rangle = 1 + \left(\frac{\rm ig}{3!}\right)^2\frac{1}{2}\int d^4x\int d^4y\,[9\,D_{xx}\,D_{xy}\,D_{yy} + 6\,D_{xy}^3] + \mathcal{O}(\lambda^3) \tag{82}$$

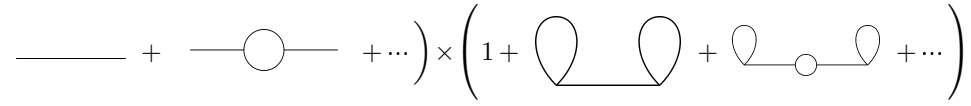
These diagrams are the bubbles  and  Expanding Eq. (79) including terms up to $\mathcal{O}(\lambda^2)$ in the numerator and denominator, we find

$$\frac{\left\langle 0\middle|T\{\,\phi_0(x_1)\,\phi_0(x_2)\,e^{i\int\mathcal{L}_{\rm int}}\}\middle|0\right\rangle}{\left\langle 0\middle|T\{e^{i\int\mathcal{L}_{\rm int}}\}\middle|0\right\rangle}=\frac{D_{12}-g^2\int\,\left[\frac{1}{8}\,D_{12}\,D_{xx}\,D_{xy}\,D_{yy}+\frac{1}{12}\,D_{12}\,D_{xy}^3+\cdots\right]}{1-g^2\int\,\left[\frac{1}{8}\,D_{xx}\,D_{xy}\,D_{yy}+\frac{1}{12}\,D_{xy}^3\right]} \tag{83}$$

Since $\frac{1}{1+g^2\,x}=1-g^2\,x+\mathcal{O}(g^4)$, we can invert the denominator in perturbation theory to see that the bubbles exactly cancel.

More generally, the bubbles will always factor out. Since the integrals in the expansion of the numerator corresponding to the bubbles never involve any external point, they just factor out. The sum over all graphs, in the numerator, is then the sum over all graphs with no-bubbles multiplying the sum over the bubbles. In pictures,

 (84)

 (85)

The sum over bubbles is exactly $\left\langle 0\middle|T\{e^{i\int\mathcal{L}_{\rm int}}\}\middle|\,0\right\rangle$. So,

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_2)\}|\Omega\rangle=\left\langle 0\middle|T\{\,\phi_0(x_1)\,\phi_0(x_2)\,e^{i\int\mathcal{L}_{\rm int}}\}\middle|0\right\rangle_{\rm no-bubbles} \tag{86}$$

where "no-bubbles" means that every connected subgraph involves an external point.

3.6 Position space Feynman rules

We have shown that the same sets of diagrams appear in the Hamiltonian approach as in the Lagrangian approach: each point x_i in the original n-point function $\langle\Omega|T\{\phi(x_1)\,\cdots\,\phi(x_n)\}|\Omega\rangle$ gets an external point and each interaction gives a new vertex whose position is integrated over and coefficient is given by the coefficient in the Lagrangian.

As long as the vertices are normalized with appropriate permutation factors, as in Eq.(27), the combinatoric factors will work out the same, as we saw in the example. In the Lagrangian approach, we saw that the coefficient of the diagram will be given by the coefficient of the interaction multiplied by the geometrical symmetry factor of the diagram. To see that this is also true for the Hamiltonian, we have to count the various combinatoric factors:

- There is a factor of $\frac{1}{m!}$ from the expansion of $\exp\,(i\,\mathcal{L}_{\rm int})=\sum\,\frac{1}{m!}(i\,\mathcal{L}_{\rm int})^m$. If we expand to order m there will be m identical vertices in the same diagram. We can also swap these vertices around, leaving the diagram looking the same. If we only include the diagram once in our final sum, the $m!$ from permuting the diagrams will cancel the $\frac{1}{m!}$ from the exponential. Neither of these factors were present in the Lagrangian approach, since internal vertices came out of splitting of lines associated with external vertices, which was unambiguous, and there was no exponential to begin with.

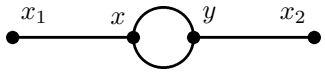
- If interactions are normalized as in Eq.(27), then there will be a $\frac{1}{j!}$ for each interaction with j identical particles. This factor is canceled by the $j!$ ways of permuting the j identical lines coming out of the same internal vertex. In the Lagrangian approach, one of the lines was already chosen so the factor was $(j-1)!$, with the missing j coming from using $\mathcal{L}'_{\rm int}[\phi]$ instead of $\mathcal{L}_{\rm int}[\phi]$.

The result is the same Feynman rules as were derived in the Lagrangian approach. In both cases, symmetry factors must be added if there is some geometric symmetry (there rarely is in theories with complex fields, like QED). In neither case do any of the diagrams include bubbles (subdiagrams which do not connect with any external vertex).

4 Momentum space Feynman rules

The position space Feynman rules derived in either of the previous two sections give a recipe for computing time-ordered products in perturbation theory. Now we will see how those time-ordered products simplify when all the phase space integrals over the propagators are performed to turn them into S-matrix elements. This will produce the momentum-space Feynman rules.

Consider the diagram

$\mathcal{T}_1 =$ 

$= -\frac{\lambda^2}{2}\int d^4x \int d^4y D_{1x} D_{xy}^2 D_{y2}$ (87)

To evaluate this diagram, first write every propagator in momentum space (taking $m = 0$ for simplicity)

$$D_{xy} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 + i\varepsilon} e^{ip(x-y)} \tag{88}$$

Then there will be four d^4p integrals from the four propagators and all the positions will appear only in exponentials. So

$$\mathcal{T}_1 = -\frac{\lambda^2}{2} \int d^4x \int d^4y \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \int \frac{d^4p_4}{(2\pi)^4} e^{ip_1(x_1-x)} e^{ip_2(y-x_2)} e^{ip_3(x-y)} e^{ip_4(x-y)} \times \frac{i}{p_1^2 + i\varepsilon} \frac{i}{p_2^2 + i\varepsilon} \frac{i}{p_3^2 + i\varepsilon} \frac{i}{p_4^2 + i\varepsilon} \tag{89}$$

Now we can do the x and y integrals, which produce $\delta^4(-p_1 + p_3 + p_4)$ and $\delta^4(p_2 - p_3 - p_4)$ respectively, corresponding to momentum being conserved at the vertices labeled x and y in the Feynman diagram. If we integrate over p_3 using the first δ-function then we can replace $p_3 = p_1 - p_4$ and the second δ-function becomes $\delta^4(p_1 - p_2)$. Then we have, relabeling $p_4 = k$:

$$\mathcal{T}_1 = -\frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} e^{ip_1x_1} e^{-ip_2x_2} \times \frac{i}{p_1^2 + i\varepsilon} \frac{i}{p_2^2 + i\varepsilon} \frac{i}{(p_1 - k)^2 + i\varepsilon} \frac{i}{k^2 + i\varepsilon} (2\pi)^4 \delta^4(p_1 - p_2) \tag{90}$$

Next, we use the LSZ theorem to convert this to a contribution to the S-matrix:

$$\langle f|S|i\rangle = \left[-i\int d^4x_1 e^{-ip_ix_1}(p_i^2)\right]\left[-i\int d^4x_2 e^{ip_fx_2}(p_f^2)\right]\langle\Omega|T\{\phi(x_1)\,\phi(x_2)\}|\Omega\rangle \tag{91}$$

where p_i^μ and p_f^μ are the initial state and final state momenta. So the contribution of this diagram gives

$$\langle f|S|i\rangle = -\int d^4x_1 e^{-ip_ix_1}(p_i)^2 \int d^4x_2 e^{ip_fx_2}(p_f^2)\,\mathcal{T}_1 + \cdots \tag{92}$$

Now we note that the x_1 integral gives $(2\pi)^4\delta^4(p_1 - p_i)$ and the x_2 integral gives a $(2\pi)^4\delta^4(p_2 - p_f)$. So we can now do the p_1 and p_2 integrals, giving

$$\langle f|S|i\rangle = -\frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p_i - k)^2 + i\varepsilon} \frac{i}{k^2 + i\varepsilon} (2\pi)^4 \delta^4(p_i - p_f) + \cdots \tag{93}$$

Note how the two propagator factors in the beginning get canceled. This always happens for external legs – remember the point of LSZ was to force the external lines to be on-shell single particle states. By the way, this integral is infinite; Part III of this text is devoted to making sense out of these infinities.

Finally, the $\delta^4(p_i - p_f)$ term in the answer forces overall momentum conservation, and will always be present in any calculation. But we will always factor it out, like we did when we related differential scattering amplitudes to S-matrix elements. Recalling that

$$\mathcal{S} = \mathbb{1} + (2\pi)^4 \delta^4(\Sigma p_i) i \mathcal{M} \tag{94}$$

and that $\mathcal{M}$ is what appears in cross sections, we have

$$i\mathcal{M} = -\frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p_i - k)^2 + i\varepsilon} \frac{i}{k^2 + i\varepsilon} + \cdots \tag{95}$$

We can summarize this procedure with the **momentum space Feynman rules**. These Feynman rules tell us how to directly calculate $i\,\mathcal{M}$ from pictures. With these rules, you can forget about anything else we've derived in this or the previous lecture. The rules are

- Internal lines (those not connected to external points) get propagators $\frac{i}{p^2 - m^2 + i\varepsilon}$.
- Vertices come from interactions in the Lagrangian. They get factors of the coupling constant times i (from the $e^{i\int \mathcal{L}_{\text{int}}}$)
- Lines connected to external points do not get propagators (their propagators are canceled by terms from the LSZ reduction formula).
- Momentum is conserved at each vertex.
- Integrate over undetermined 4-momenta.

Then one sums over all possible diagrams. The combinatoric factor for the diagram, as contributing to the momentum space Feynman rules, is given only by the geometric symmetry factor of the diagram.

Identical particles are already taken care of in Wick's theorem; moving around the a_p's and $a_p^\dagger$'s has the algebra of identical particles in them. The only time identical particles need extra consideration is when we cannot distinguish the particles we are scattering. This only happens for final states, since we distinguish our initial states by the setup of the experiment. Thus when n of the same particles are produced, we have to divide the cross-section by $n!$.

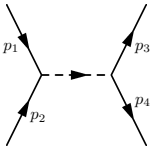
4.1 Signs of momenta

There is unfortunately no standard convention about how to choose the direction that the momenta are going. For external momenta it makes sense to assign them their physical values, which should have positive energy. Then momentum conservation becomes

$$\sum p_i = \sum p_f \tag{96}$$

which appears in a δ-functions as $\delta^4(\sum p_i - \sum p_f)$.

For internal lines, we integrate over the momenta, so it doesn't matter if we use k_μ or $-k_\mu$. Still, it is important to keep track of which way the momentum is going so that all the δ-functions at the vertices are also $\Sigma(p_{\text{in}} - p_{\text{out}})$. So we draw arrows on the lines to indicate this



Actually, the arrows will eventually come to be associated with the momentum direction only for particles with antiparticles will moving backwards to the direction of arrows. We will sometimes add additional arrows to refer to just the momentum for additional clarity.

You should be warned that sometimes Feynman diagrams are drawn with time going upwards, particularly in describing hadronic collisions.

4.2 Disconnected graphs

A lot of the contractions will result in diagrams where some subset of the the external vertices connect to each other without interacting with the other subsets.

What about graphs where subsets are independently connected, such as the contribution to the 8-point function shown on the left in Figure 1? Diagrams like this have physical effects. For example, at a muon collider, there would be a contribution to the S-matrix from situations where the muons just decay independently, somewhat close to the interaction region, which look like the left graph, in addition to the contribution where the muons scatter off of each other, which might look like the right graph in Figure 1.

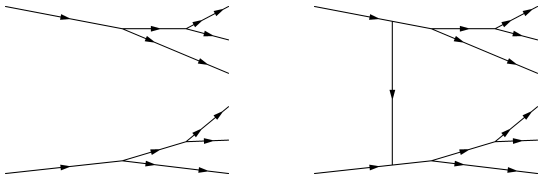


Figure 1. Disconnected graphs like the one on the left have important physical effects. However, they have a different singularity structure and therefore zero interference.

Clearly both processes need to be incorporated for an accurate description of the collision. However, the disconnected decay process can be computed from the S-matrix for $1 \to 3$ scattering (as in either half of the left diagram). The probability for the $2 \to 6$ process from the disconnected diagram is then just the product of the two $1 \to 3$ probabilities. More generally, the S-matrix with bubbles removed factorizes into a product of sums of connected diagrams, just like the bubbles factorized out of the full S-matrix.

The only possible complication is if there could be interference between the disconnected diagrams and the connected ones. However, this cannot happen: there is zero interference. To see why, recall that the definition of the matrix element which these time-ordered calculations produce has only a single δ-function

$$\mathcal{S} = \mathbb{1} + i\delta^4(\Sigma\, p)\mathcal{M} \tag{97}$$

The disconnected matrix elements will have an extra δ-function $\mathcal{M}_{\text{disconnected}} = \delta^4(\Sigma_{\text{subset}}\, p)(\cdots)$ and therefore cannot interfere with the connected amplitude $\mathcal{M}_{\text{connected}}$, which is just some integral over propagators, as given by the Feynman rules. Such an integral can only have poles or possibly branch cuts, but is an analytic function of the external momenta away from these. It can never produce singularities as strong as δ-functions. (The same decoherence is also relevant for meta-stable particles produced in collisions, where it leads to the narrow-width approximation, to be discussed in Lecture III-12.) You can check this in Problem 3.

More profoundly, that there can never be more than a single δ-function coming out of connected amplitudes is related to a general principle, which Weinberg takes as an axiom of quantum field theory, called **cluster decomposition**. The cluster decomposition principle says that experiments well-separated in space cannot influence each other. More precisely, as positions in one subset become well-separated from positions in the other subsets, the connected S-matrix should vanish. If there were an extra δ-function, one could asymptotically separate some of the points in such a way that the S-matrix went to a constant, violating cluster decomposition. Weinberg emphasizes that constructing local theories out of *fields* made from creation and annihilation operators guarantees cluster decomposition, as we have seen. However, it is not known whether the logic is invertible, that is, if the only possible theories which satisfy cluster decomposition are local field theories constructed out of creation and annihilation operators. It is also not clear how well cluster decomposition has been tested experimentally.

Technicalities of cluster decomposition aside, the practical result of this section is that the only thing we ever need to compute for scattering processes is

$$\langle 0|T\{\phi(x_1)\cdots\phi_0(x_n)\}|\,0\rangle_{\text{connected}} \tag{98}$$

where "connected" means every external vertex connects to every other external vertex through the graph somehow. Everything else is factored out or normalized away. Bubbles come up occasionally in discussions of vacuum energy; disconnected diagrams are never important.

5 Examples

The Feynman rules will all make a lot more sense after we do some examples. Let us start with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\phi\Box\phi - \frac{1}{2}\,m^2\,\phi^2 + \frac{g}{3!}\,\phi^3 \tag{99}$$

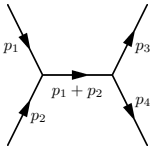
and consider the differential cross section for $\phi\,\phi \to \phi\,\phi$ scattering. In the center of mass frame, the cross section is related to the matrix element by Eq. ?? from Lecture 1-5:

$$\frac{d\sigma}{d\Omega}(\phi\,\phi \to \phi\,\phi) = \frac{1}{64\,\pi^2\,E_{\rm cm}^2}|\mathcal{M}|^2 \tag{100}$$

Let the incoming momenta be p_1^μ and p_2^μ and the outgoing momenta be p_3^μ and p_4^μ.

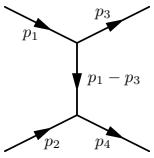
There are 3 diagrams. The first gives

$$i\,\mathcal{M}_s = \quad = (ig)\frac{i}{(p_1+p_2)^2 - m^2 + i\varepsilon}(ig) = \frac{-ig^2}{s - m^2 + i\varepsilon}$$



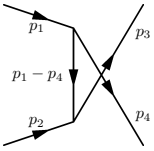
where $s \equiv (p_1 + p_2)^2$. The second gives

$$i\,\mathcal{M}_t = \quad = (ig)\frac{i}{(p_1-p_3)^2 - m^2 + i\varepsilon}(ig) = \frac{-ig^2}{t - m^2 + i\varepsilon} \tag{101}$$



where $t \equiv (p_1 - p_3)^2$. And the final diagram evaluates to

$$i\,\mathcal{M}_3 = \quad = (ig)\frac{i}{(p_1-p_4)^2 - m^2 + i\varepsilon}(ig) = \frac{-ig^2}{u - m^2 + i\varepsilon} \tag{102}$$



where $u \equiv (p_1 - p_4)^2$. The sum is

$$\frac{d\sigma}{d\Omega}(\phi\,\phi \to \phi\,\phi) = \frac{1}{2}\,\frac{g^4}{64\,\pi^2\,E_{\rm cm}^2}\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2 \tag{103}$$

with the $\frac{1}{2}$ coming from identical particles. We have dropped the $i\varepsilon$, which is fine as long as s, t, u are not equal to m^2. (For that to happen, the intermediate scalar would have to go on-shell in one of the diagrams, which is a degenerate situation, usually contributing only to $\mathbb{1}$ in the S-matrix. The $i\varepsilon$'s will be necessary for loops, but in tree-level diagrams you can pretty much ignore them.)

5.1 Mandelstam variables

The variables s, t, u are called **Mandelstam variables**. They are a great shorthand, used almost exclusively in $2 \to 2$ scattering and in $1 \to 3$ decays, although there are generalizations for more momenta. For $2 \to 2$ scattering, with initial momenta p_1 and p_2 and final momenta p_3 and p_4, they are defined by

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{104}$$

$$t \equiv (p_1 - p_3)^2 = (p_2 - p_4)^2 \tag{105}$$

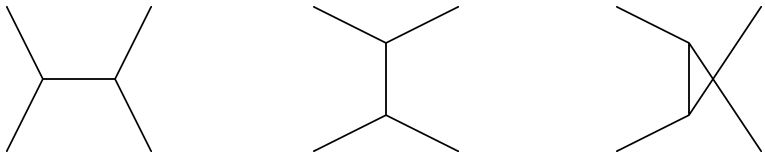
$$u \equiv (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{106}$$

These satisfy

$$s + t + u = \sum m_j^2 \tag{107}$$

where m_j are the invariant masses of the particles.

As we saw in the previous example, s, t and u correspond to particular diagrams where momentum in the propagator has invariant $p_\mu^2 = s$, t or u. We say s-channel for annihilation diagrams. In these the intermediate state has $p_\mu^2 = s > 0$. The t- and u- channels are scattering diagrams



$s-$channel $t-$channel $u-$channel

s, t and u are great because they are Lorentz invariant. So we compute $\mathcal{M}(s,t,u)$ in the center-of-mass frame, and then we can easily find out what it is in any other frame, for example the frame of the lab in which we are doing the experiment. We will use s, t and u a lot.

5.2 Derivative couplings

Suppose we have an interaction with derivatives in it, like

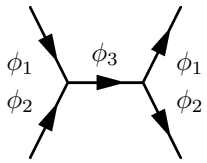
$$\mathcal{L}_{\rm int} = \lambda\,\phi_1(\partial_\mu\phi_2)(\partial_\mu\phi_3) \tag{108}$$

where I have included 3 different scalar fields for clarity. In momentum space, these ∂_μ's give factors of momenta. But now remember that

$$\phi(x) = \int \frac{d^3\,p}{(2\,\pi)^3}\frac{1}{\sqrt{2\,\omega_p}}\big(a_p\,e^{-ipx} + a_p^\dagger\,e^{ipx}\big) \tag{109}$$

So if the particle is being created (emerging from a vertex) it gets a factor of ip_μ, and if it's being destroyed (entering a vertex) it gets a factor of $-ip_\mu$. So a $-$ for incoming momentum and a $+$ for outgoing momentum. In this case, it's quite important to keep track of whether momentum is flowing into or out of the vertex.

For example, take the diagram



(110)

Label the initial momenta p_1^μ and p_2^μ and the final momenta $p_1^{\mu\prime}$ and $p_2^{\mu\prime}$. The exchanged momentum is $k^\mu = p_1^\mu + p_2^\mu = p_1^{\mu\prime} + p_2^{\mu\prime}$. Then this diagram gives

$$i\mathcal{M} = (i\,\lambda)^2(-ip_2^\mu)(ik^\mu)\frac{i}{k^2}(ip_2^{\nu\prime})(-ik^\nu) = -i\,\lambda^2\,\frac{[p_2\cdot p_1 + (p_2)^2][p_2'\cdot p_1' + (p_2')^2]}{(p_1+p_2)^2} \tag{111}$$

As a cross check, we should get the same answer if we use a different Lagrangian related to the one we used by integration by parts:

$$\mathcal{L}_{\rm int} = -\lambda\,\phi_3[(\partial_\mu\phi_1)(\partial_\mu\phi_2) + \phi_1\Box\phi_2] \tag{112}$$

Now our one diagram becomes four diagrams, from the two types of vertices on the two sides, all of which look like Eq. (110). It's easiest to add up the contributions to the vertices before multiplying, which gives

$$\mathcal{M} = (i\,\lambda)^2[(-ip_2^\mu)(-ip_1^\mu) + (-ip_2)^2]\,\frac{i}{k^2}[(ip_2^{\nu\prime})(ip_1'^{\,\nu}) + (ip_2')^2] \tag{113}$$

$$= -i\,\lambda^2\,\frac{[p_2\cdot p_1 + (p_2)^2][p_2'\cdot p_1' + (p_2')^2]}{(p_1+p_2)^2} \tag{114}$$

which is exactly what we had above. So integrating by parts does not affect the matrix elements, as expected. Thus the Feynman rules passed our cross check.

To see more generally that integrating by parts does not affect matrix elements, it is enough to show that total derivatives do not contribute to matrix elements. Suppose we have a term

$$\mathcal{L}_{\rm int} = \partial_\mu(\phi_1 \cdots \phi_n) \tag{115}$$

where there are any number of fields in this term. This would give a contribution from the derivative acting on each field, with a factor of that field's momenta. So if the vertex would have given V without the derivative, adding the derivative makes it

$$\left(\sum_{\rm incoming} p^i_\mu - \sum_{\rm outgoing} p^j_\mu\right) V \tag{116}$$

Since the sum of incoming momenta is equal to the sum of outgoing momenta, because momentum is conserved at each vertex, we conclude that total derivatives do not contribute to matrix elements

To be precise, total derivatives do not contribute to matrix elements *in perturbation theory*. It turns out a term like

$$\tilde{F}F \equiv \varepsilon_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = 4\partial_\mu(\varepsilon_{\mu\nu\alpha\beta} A_\alpha \partial_\beta A_\nu) \tag{117}$$

is a total derivative. If we add a term $\theta\, \tilde{F}\, F$ to the Lagrangian, indeed nothing happens in perturbation theory. It turns out that there are effects of this term that will never show up in Feynman diagrams, but are perfectly real. They have physical consequences. For example, if this term appeared in the Lagrangian with anything but an exponentially small coefficient, it would lead to an observable electric dipole moment for the neutron. That no such moment has been seen is known as the strong CP problem (see Lecture IV-4). A closely related effect from such a total derivative is the mass of the η' meson, which is larger than could be possible without total-derivative terms (see Lecture IV-6). In the η' meson case, the mass comes from the strong interactions which are non-perturbative.

Appendix A Normal ordering and Wick's theorem

In this Appendix we prove that that the vacuum matrix element of a time-ordered product of free fields is given by the sum of all possible full contractions, a result known as Wick's theorem. This theorem is necessary for the derivation of the Feynman rules in the Hamiltonian approach.

A.1 Normal Ordering

To prove Wick's theorem, we will manipulate expressions with creation and annihilation operators into the form of a c-number expression plus terms which vanish when acting on the vacuum. This is always possible since we can commute the annihilation operators past the creation operators until they are all on the right at which point they give zero when acting on the vacuum.

For example, we can write

$$(a^\dagger_p + a_p)(a^\dagger_k + a_k) = [a_p, a^\dagger_k] + a^\dagger_k a_p + a^\dagger_p a_k + a_p a_k + a^\dagger_p a^\dagger_k \tag{118}$$

$$= (2\pi)^3 \delta^3(p-k) + a^\dagger_k a_p + a^\dagger_p a_k + a_p a_k + a^\dagger_p a^\dagger_k \tag{119}$$

Then, since the terms with annihilation operators on the right vanish, as do the terms with creation operators on the left, we get

$$\left\langle 0 \right| (a^\dagger_p + a_p)(a^\dagger_k + a_k) \left| 0 \right\rangle = (2\pi)^3 \delta^3(p-k) \tag{120}$$

We call terms with all annihilation operators on the right *normal-ordered.*

- **Normal-ordered**: all the $a^\dagger_p$ operators are on the left of all the a_p operators.

We represent normal ordering with colons. So

$$:(a_p^\dagger + a_p)(a_k^\dagger + a_k): = a_k^\dagger a_p + a_p^\dagger a_k + a_p a_k + a_p^\dagger a_k^\dagger \tag{121}$$

When you normal-order something, you just pick up the operators and move them. Just manhandle them over, without any commuting, just as you manhandled the operators within a time-ordered product. Thus the $\delta(p-k)$ from Eq. (119) does not appear in Eq. (121).

The point of normal-ordering is that vacuum matrix elements of normal ordered products of fields vanish

$$\langle 0|: \phi(x_1) \cdots \phi(x_n): |0\rangle = 0 \tag{122}$$

The only normal-ordered expressions which don't vanish in the vacuum are c-number functions. Such a function f satisfies

$$\langle 0|: f :|0\rangle = f \tag{123}$$

The nice thing about normal-ordering is that we can use it to specify operator relations. For example,

$$T\{\phi_0(x)\,\phi_0(y)\} =: \phi_0(x)\,\phi_0(y) + D_F(x,y): \tag{124}$$

This is obviously true in vacuum matrix elements, since $D_F(x,\,y) = \langle 0|T\{\phi_0(x)\;\phi_0(y)\}|\;0\rangle$ and vacuum matrix elements of normal-ordered products vanish. But it is also true at the level of the operators, as we show below. The point is that by normal-ordering expressions we can read off immediately what will happen when we take vacuum matrix elements, but no information is thrown out.

A.2 Wick's theorem

Wick's theorem says that

$$T\{\phi_0(x_1) \cdots \phi_0(x_n)\} =: \phi_0(x_1) \cdots \phi_0(x_n) + \begin{array}{c}\text{all possible}\\ \text{contractions}\end{array} : \tag{125}$$

where a contraction means take two fields $\phi_0(x_i)$ and $\phi_0(x_j)$ from anywhere in the series and replace them with a factor of $D_F(x_i, x_j)$ for each pair of fields. All possible contractions includes one contraction, two contractions, etc., involving any of the fields. But each field can only be contracted once. Since normal-ordered products vanish unless all the fields are contracted this implies that the time-ordered product is the sum of all the full contractions, which is what we'll actually use to generate Feynman rules.

Wick's theorem is easiest to prove first by breaking the field up into creation and annihilation parts, $\phi_0(x) = \phi_+(x) + \phi_-(x)$ where

$$\phi_+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\,\omega_p}}\,a_p^\dagger\, e^{ipx}, \quad \phi_-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\,\omega_p}}\,a_p\, e^{-ipx} \tag{126}$$

Since $[a_k,\, a_p^\dagger] = (2\pi)^3\,\delta^3(\vec{p}\,-\,\vec{k}\,)$ commutators of these operators are just functions. In fact, the Feynman propagator can be written as

$$\begin{aligned} D_F(x_1,x_2) &= \langle 0|T\{\phi_0(x_1)\,\phi_0(x_2)\}|\,0\rangle \\ &= \big[\,\phi_-\big(x_1\big),\,\phi_+(x_2)\,\big]\theta(t_1-t_2) + [\phi_-(x_2),\,\phi_+(x_1)]\theta(t_2-t_1) \end{aligned}$$

This particular combination represents a contraction.

Let's verify Wick's theorem for 2 fields. For $t_1 > t_2$

$$T\{\phi_0(x_1)\,\phi_0(x_2)\} = \phi_+(x_1)\,\phi_+(x_2) + \phi_+(x_1)\,\phi_-(x_2) + \phi_-(x_1)\,\phi_+(x_2) + \phi_-(x_1)\,\phi_-(x_2) \tag{127}$$

All terms in this expression are normal ordered except for $\phi_-(x_1)\,\phi_+(x_2)$. So,

$$T\{\phi_0(x_1)\,\phi_0(x_2)\} =: \phi_0(x_1)\,\phi_0(x_2): + [\phi_-(x_1),\,\phi_+(x_2)], \quad t_1 > t_2 \tag{128}$$

For $t_2 > t_1$, the expression is the same with $x_1 \leftrightarrow x_2$. Thus

$$T\{\phi_0(x_1)\,\phi_0(x_2)\} =: \phi_0(x_1)\,\phi_0(x_2): + D_F(x_1,x_2) \tag{129}$$

exactly as Wick's theorem requires.

The full proof is straightforward to do by mathematical induction. We have shown it works for 2 fields. Assume it holds for $n-1$ fields. Without loss of generality, let t_1 be the latest time for all n fields. Then

$$T\{\phi_0(x_1)\,\phi_0(x_2)\cdots\phi(x_n)\} = \big[\,\phi_+\big(x_1\big) + \phi_-(x_1)\,\big]\mathbf{:}\,\phi_0(x_2)\cdots\phi_0(x_n) + \begin{matrix}\text{all possible}\\ \text{contractions}\end{matrix}\ \mathbf{:} \tag{130}$$

Since $\phi_+(x_1)$ is on the left and contains $a_p^\dagger$ operators, we can just move it into the normal ordering. The $\phi_-(x_1)$ must be moved through to the right. Each time it passes a $\phi_+(x_i)$ field in the normal ordered product, a contraction results. The result is the sum over the normal ordered product of n fields and all possible contractions of $\phi_-(x_1)$ with any of the $\phi_+(x_i)$ in any of the terms in the normal ordered product in Eq. (130). That is exactly what all possible contractions of the fields $\phi_0(x_2)$ to $\phi_0(x_n)$ means. Thus Wick's theorem is proven.

The result of Wick's theorem is that time-ordered products are given by a bunch of contractions plus normal ordered products. Since the normal-ordered products vanish in vacuum matrix elements, all that remains for vacuum matrix elements of time ordered products are the Feynman propagators.