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III-3: The anomalous magnetic moment

1 Introduction

In the non-relativistic limit, the Dirac equation in the presence of an external magnetic field produces a Hamiltonian

$$H = \frac{\vec{p}^{\,2}}{2m} + V(r) + \frac{e}{2m}\vec{B}\cdot(\vec{L} + g\,\vec{S}) \tag{1}$$

acting on electron doublets $|\psi\rangle$ where $\vec{S} = \frac{1}{2}\vec{\sigma}$. This was derived in Problem ?? of Lecture ??. The coupling g is the ***g*-factor** of the electron, representing the relative strength of its intrinsic magnetic dipole moment to the strength of the spin-orbit coupling. From the point of view of the Schrödinger equation, g is a free parameter and could be anything. However, the Dirac equation implies that $g = 2$ which was an historically important postdiction in excellent agreement with data when Dirac presented his equation in 1932. A natural question is then, is $g = 2$ exactly, or does g receive quantum corrections? The answer should not be obvious. For example, the charge of the electron is *exactly* opposite the charge of the proton, receiving no radiative corrections (we will prove this in Lecture III-5), so perhaps the magnetic moment is exact as well. By the late 1940s there was experimental data which could be partially explained by the electron having an **anomalous magnetic moment**, that is, one different from 2. The calculation of this anomalous moment by Schwinger, Feynman and Tomonaga in 1948, and its agreement with data, was a triumph of quantum field theory.

2 Extracting the moment

We would like a way to extract the radiative corrections to g without having to take the non-relativistic limit. To see how to do this recall from Lecture II-3 how the electron's magnetic dipole moment was derived from the Dirac equation. Charged spinors satisfy $(i\not{D} - m)\psi = 0$. Multiplying this by $(i\not{D} + m)$ shows that charged spinors also satisfy $\left(\not{D}^2 + m^2\right)\psi = 0$. We then use the operator relation (cf. Eq. (109??) of Lecture II-3)

$$\not{D}^2 = D_\mu^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} \tag{2}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ to find $\left(D_\mu^2 + m^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}\right)\psi = 0$. The $\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}$ in this equation therefore encodes the difference between the way a scalar field, obeying $(D_\mu^2 + m^2)\phi = 0$, and a spinor field interact with an electromagnetic field. In particular, in the Weyl representation

$$\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} = -e\begin{pmatrix} (\vec{B} + i\vec{E})\vec{\sigma} & \\ & (\vec{B} - i\vec{E})\vec{\sigma} \end{pmatrix} \tag{3}$$

Going to momentum space $\left(\not{D}^2 + m^2\right)\psi = 0$ implies (cf. Eq. (112??) of Lecture II-3),

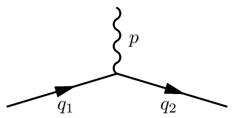
$$\frac{(H - eA_0)^2}{2m}\psi = \left(\frac{m}{2} + \frac{\left(\vec{p} - e\vec{A}\,\right)^2}{2m} - 2\,\frac{e}{2m}\vec{B}\cdot\vec{S} \pm i\frac{e}{m}\,\vec{E}\cdot\vec{S}\right)\psi \tag{4}$$

which can be compared directly to Eq. (1) to read off the strength of the magnetic dipole interaction $ge\vec{B}\cdot\vec{S}$.[1] Since $\vec{S} = \frac{\vec{\sigma}}{2}$ for spin $\frac{1}{2}$, we find again that $g = 2$. If Eq. (2) had $g'\frac{e}{4}F_{\mu\nu}\sigma^{\mu\nu}$ in it, we would have found $g = g'$ instead. Thus a general and relativistic way to extract corrections to g is to look for loops which have the same effect as an additional $F_{\mu\nu}\sigma^{\mu\nu}$ term.

1. The $\vec{E}\cdot\vec{S}$ term is not an electric dipole moment since it has an imaginary coefficient. Instead, it is Lorentz invariant completion of the magnetic moment.

A generally useful way to think about corrections to the way photons interact with spinors, such as corrections to g, is to consider *off-shell* S-matrix elements. The Feynman rules for off-shell S-matrix elements are the same as for on-shell S-matrix elements, except that $p_i^2 = m_i^2$ for the various external states is not enforced. In this case, the relevant process is $e^-(q_1)A_\mu(p) \to e^-(q_2)$ with polarization vector $\epsilon_\mu(p)$ and two spinor states $\bar{u}(q_2)$ and $u(q_1)$. At tree level, the matrix element is just $\epsilon_\mu \mathcal{M}_0^\mu$ where

$$i\mathcal{M}_0^\mu =$$



$$= -ie\bar{u}(q_2)\gamma^\mu u(q_1) \qquad (5)$$

with the photon momentum constrained by momentum conservation to be $p^\mu = q_2^\mu - q_1^\mu$. This result actually contains $g = 2$ in it, although it is hard to see in this form. We expect something equivalent to an $F_{\mu\nu}\sigma^{\mu\nu}$ term which should look like $\bar{u}(q_2)p_\nu\sigma^{\mu\nu}u(q_1)$ in momentum space. To see where $F_{\mu\nu}\sigma^{\mu\nu}$ is hiding, we need to massage the result a little.

For the magnetic moment, we only have to allow for the photon, which corresponds to an unconstrained external magnetic field, to be off-shell; the spinors can be on-shell, which helps simplify things. For example, we can use the **Gordon identity**, which you derived in Problem ??, and holds for on-shell spinors:

$$\bar{u}(q_2)(q_1^\mu + q_2^\mu)u(q_1) = (2m)\,\bar{u}(q_2)\gamma^\mu u(q_1) + i\,\bar{u}(q_2)\sigma^{\mu\nu}(q_1^\nu - q_2^\nu)u(q_1) \qquad (6)$$

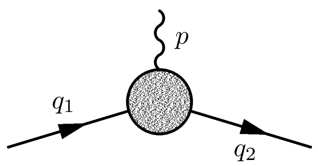
Therefore

$$\mathcal{M}_0^\mu = -e\left(\frac{q_1^\mu + q_2^\mu}{2m}\right)\bar{u}(q_2)u(q_1) - \frac{e}{2m}\,i\,\bar{u}(q_2)p_\nu\sigma^{\mu\nu}u(q_1) \qquad (7)$$

The first term is an interaction just like the scalar QED interaction: the photon couples to the momentum of the field, as in the D_μ^2 term in the Klein-Gordon equation. The q_μ^1 and q_μ^2 in this first term are just the momentum factors which appear in the scalar QED Feynman rule. The second term in Eq. (7) is spin-dependent and gives the magnetic moment. So we can identify g as $\frac{4m}{e}$ times the coefficient of $ip_\nu\bar{u}\sigma^{\mu\nu}u$. Therefore, to calculate corrections to g we need to find how the coefficient of $\bar{u}p_\nu\sigma^{\mu\nu}u$ is modified at loop level.

The correction to the magnetic moment must come from graphs involving the photon and the electron, to contribute corrections to the process in Eq. (5). We can parameterize the most general possible result, at any-loop order, as

$$i\mathcal{M}^\mu =$$



$$= \bar{u}(q_2)(f_1\gamma^\mu + f_2 p^\mu + f_3 q_1^\mu + f_4 q_2^\mu)u(q_1) \qquad (8)$$

Here we have included all Lorentz vectors which might possibly appear, with the f_i their unknown Lorentz scalar coefficients. The f_i can depend in general on contractions of momenta, like $p \cdot q$ or p^2, or on contractions with γ-matrices like $\not{p}$. (In more general theories, they could also depend on γ_5, but QED is parity invariant so γ_5 cannot appear.) For the magnetic moment application, we can assume the external spinors are on-shell, but the photon, representing an unconstrained external magnetic field, must still be off-shell. (Or, if you prefer, imagine this diagram is embedded in a larger Coulomb-scattering diagram with an off-shell intermediate photon and on-shell external spinors.)

The f_i are not all independent. Using momentum conservation, $p^\mu = q_2^\mu - q_1^\mu$, we can set $f_2 = 0$ and substitute away all the p^μ dependence. Then, if there are factors of $\not{q}_1$ or $\not{q}_2$ in the f_i, they can be removed by using the Dirac equation $\not{q}_1 u(q_1) = m\,u(q_1)$ and $\bar{u}(q_2)\not{q}_2 = m\bar{u}(q_2)$. So, we can safely assume the f_i are real functions which can only depend on $q_1 \cdot q_2$ and m, or more conventionally on $p^2 = 2m^2 - 2q_1 \cdot q_2$ and m^2. Moreover, we can fix the relative dependence by dimensional analysis so the f_i are functions of $\frac{p^2}{m^2}$.

Next, the Ward identity (which we showed in Lecture II-7 holds even if the photon is off-shell) implies

$$\begin{aligned} 0 &= p_\mu \bar{u}\,(f_1\gamma^\mu + f_3 q_1^\mu + f_4 q_2^\mu)u \\ &= f_1 \bar{u}\,\not{p}\,u + (p\cdot q_1) f_3 \bar{u}u + (p\cdot q_2) f_4 \bar{u}u \\ &= (p\cdot q_1) f_3 \bar{u}u + (p\cdot q_2) f_4 \bar{u}u \end{aligned} \tag{9}$$

We then use $p\cdot q_1 = q_2\cdot q_1 - m^2 = -p\cdot q_2$ to get $f_3 = f_4$. Thus there are only two independent form factors. We can then use the Gordon identity, Eq. (6), to rewrite the q_1^μ and q_2^μ dependence in terms of $\sigma^{\mu\nu}$, leading to

$$i\mathcal{M}^\mu = (-ie)\bar{u}\,(q_2)\left[F_1\left(\frac{p^2}{m^2}\right)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m} p_\nu F_2\left(\frac{p^2}{m^2}\right)\right]u(q_1) \tag{10}$$

which is our final form. This parametrization holds to all orders in perturbation theory. The functions F_1 and F_2 are known as **form factors**. The leading graph, Eq. (5) gives

$$F_1 = 1, \quad F_2 = 0 \tag{11}$$

Loops will give contributions to F_1 and F_2 at order α and higher.

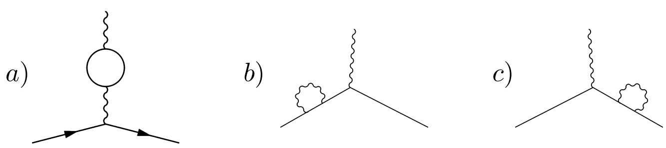
Which of these two form factors could give an electron magnetic moment? F_1 modifies the original $e A_\mu \bar{\psi}\gamma^\mu\psi$ coupling. This renormalizes the electric charge, as we saw from the vacuum-polarization diagram. In fact, the entire effect of this form-factor is to give scale-dependence to the electric charge, so no other effect, like an anomalous magnetic moment, can come from it. F_2, on the other hand, has precisely the structure of a magnetic moment (which is, of course, why we put it in this form with the Gordon identity). Using that such a term without the F_2 factor gives $g = 2$, as in Eq. (7), we conclude that $F_2\left(\frac{p^2}{m^2}\right)$ modifies the moment at the scale associated with p^2 by $g \to 2 + 2F_2\left(\frac{p^2}{m^2}\right)$. Since the actual magnetic moment is measured at non-relativistic energies with $|\vec{p}\,| \ll m$, the moment which can be compared to data is

$$g = 2 + 2F_2(0) \tag{12}$$

Thus, we have reduced the problem to calculating $F_2(0)$.

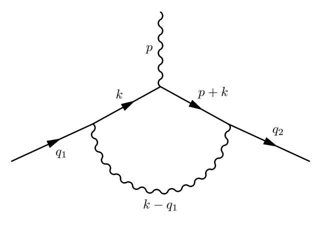
3 Evaluating the graphs

There are 4 possible 1-loop graphs which could contribute to $\mathcal{M}^\mu$. Three of them,



can only give terms proportional to γ^μ. This is easy to see because these graphs just correct the propagators for the corresponding particles. Thus these graphs can only contribute to F_1 and have no effect on the magnetic moment. The fourth graph is

$i\mathcal{M}_2^\mu =$



with $p^\mu = q_2^\mu - q_1^\mu$. This is the only graph we have to consider for $g-2$.

Employing the Feynman rules, this graph is

$$i\mathcal{M}_2^\mu = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\nu\alpha}}{(k-q_1)^2 + i\varepsilon} \bar{u}(q_2)\gamma^\nu \frac{i(\not p + \not k + m)}{(p+k)^2 - m^2 + i\varepsilon} \gamma^\mu \frac{i(\not k + m)}{k^2 - m^2 + i\varepsilon} \gamma^\alpha u(q_1) \tag{13}$$

$$= -e^3 \bar{u}(q_2) \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu(\not p + \not k + m)\gamma^\mu(\not k + m)\gamma_\nu}{[(k-q_1)^2 + i\varepsilon][(p+k)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]} u(q_1) \tag{14}$$

To simplify this, we start by combining denominators and completing the square. The denominator has 3 terms and can be simplified with the identity

$$\frac{1}{ABC} = 2\int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{1}{[xA + yB + zC]^3} \tag{15}$$

In this case

$$\begin{aligned} A &= k^2 - m^2 + i\varepsilon \\ B &= (p+k)^2 - m^2 + i\varepsilon \\ C &= (k - q_1)^2 + i\varepsilon \end{aligned}$$

The new denominator is the cube of

$$\begin{aligned} xA + yB + zC &= k^2 + 2k(yp - zq_1) + yp^2 + zq_1^2 - (x+y)m^2 + i\varepsilon \\ &= (k^\mu + yp^\mu - zq_1^\mu)^2 - \Delta + i\varepsilon \end{aligned}$$

with

$$\Delta = -xyp^2 + (1-z)^2 m^2 \tag{16}$$

Thus we want to shift $k^\mu \to k^\mu - yp^\mu + zq_1^\mu$ to make the denominator $(k^2 - \Delta)^3$.

The numerator in Eq. (14) is

$$\begin{aligned} N^\mu &= \bar{u}(q_2)\gamma^\nu(\not p + \not k + m)\gamma^\mu(\not k + m)\gamma_\nu u(q_1) \\ &= -2\bar{u}(q_2)[\not k\gamma^\mu \not p + \not k \gamma^\mu \not k + m^2\gamma^\mu - 2m(2k^\mu + p^\mu)]u(q_1) \end{aligned}$$

Shifting $k^\mu \to k^\mu - yp^\mu + zq_1^\mu$ then gives

$$-\frac{1}{2}N^\mu = \bar{u}(q_2)[(\not k - y\not p + z\not q_1)\gamma^\mu \not p + (\not k - y\not p + z\not q_1)\gamma^\mu(\not k - y\not p + z\not q_1)]u(q_1) \tag{17}$$

$$+ \bar{u}(q_2)[m^2\gamma^\mu - 2m(2k^\mu - 2yp^\mu + 2zq_1^\mu + p^\mu)]u(q_1) \tag{18}$$

Using $k^\mu k^\nu = \frac{1}{4}g^{\mu\nu}k^2$, the Gordon identity, $x + y + z = 1$ and a fair amount of algebra, this simplifies to

$$\begin{aligned} -\frac{1}{2}N^\mu = &\left[-\frac{1}{2}k^2 + (1-x)(1-y)p^2 + (1 - 4z + z^2)m^2\right]\bar{u}(q_2)\gamma^\mu u(q_1) \\ &+ imz(1-z)p_\nu \bar{u}(q_2)\sigma^{\mu\nu}u(q_1) \\ &+ m(z-2)(x-y)p^\mu \bar{u}(q_2)u(q_1) \end{aligned} \tag{19}$$

We have found 3 independent terms instead of 2 since we have not used the Ward identity. Indeed the Ward identity should fall out of the calculation automatically. To see that it does, note that the p^μ term gives a contribution to $\mathcal{M}_2^\mu$ of the form

$$i\mathcal{M}_2^\mu = 4e^3 \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) m(z-2)(x-y) \int \frac{d^4k}{(2\pi)^4} \frac{p^\mu}{(k^2 - \Delta + i\varepsilon)^3} \bar{u}(q_2)u(q_1) \tag{20}$$

Next, note that both Δ in Eq.(16) and the integral measure are symmetric in $x \leftrightarrow y$, but the integrand is antisymmetric. Thus this term is zero.

For the magnetic moment calculation we only need the $\sigma^{\mu\nu}$ term. Thus

$$i\mathcal{M}_2^\mu = p_\nu \bar{u}(q_2)\sigma^{\mu\nu}u(q_1)\left[4ie^3 m \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4} \frac{z(1-z)}{(k^2 - \Delta + i\varepsilon)^3}\right] + \cdots \tag{21}$$

where the $\cdots$ do not contribute to the moment. Recalling that $F_2(p^2)$ was defined as the coefficient of this operator, normalized by $\frac{2\,m}{e}$, we have

$$F_2(p^2) = \frac{2m}{e}(4ie^3m)\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int \frac{d^4k}{(2\pi)^4}\frac{z(1-z)}{(k^2-\Delta+i\varepsilon)^3} + \mathcal{O}(e^4) \tag{22}$$

For completeness, the other form factor is $F_1(p^2) = 1 + f(p^2) + \mathcal{O}(e^4)$ where

$$f(p^2) = -2ie^2\int_0^1 \frac{d^4k}{(2\pi)^4} dx\,dy\,dz\,\delta(x+y+z-1)\frac{k^2 - 2(1-x)(1-y)p^2 - 2(1-4z+z^2)m^2}{[k^2 - (m^2(1-z)^2 - xyp^2)]^3} \tag{23}$$

We will come back and evaluate $f(p^2)$ when we need to (in Lecture III-5).

To evaluate F_2, we use the identity from Appendix B

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\varepsilon)^3} = \frac{-i}{32\pi^2\Delta} \tag{24}$$

to get that, up to terms of order α^2,

$$F_2(p^2) = \frac{\alpha}{\pi}m^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{z(1-z)}{(1-z)^2m^2 - xyp^2} \tag{25}$$

At $p^2 = 0$ this integral is finite. Explicitly

$$\begin{aligned} F_2(0) &= \frac{\alpha}{\pi}\int_0^1 dz\int_0^1 dy\int_0^1 dx\,\delta(x+y+z-1)\frac{z}{(1-z)} \\ &= \frac{\alpha}{\pi}\int_0^1 dz\int_0^{1-z} dy\,\frac{z}{(1-z)} \\ &= \frac{\alpha}{2\pi} \end{aligned}$$

Thus

$$\boxed{g = 2 + \frac{\alpha}{\pi} = 2.00232} \tag{26}$$

with the next correction of order α^2.

As a historical note, this result was first announced at the APS meeting in January 1948, by Schwinger. Feynman and Tomonaga had both calculated the same result independently at the same time. Schwinger actually found a different value for $g-2$ for an electron bound in an atom and a free electron, while Feynman found they were the same. Feynman's result was the correct one, and it was relativistically invariant, while Schwinger's was not. The discrepancy was quickly resolved. Tomonaga was the first to correctly present the full 1-loop formula for the Lamb shift.

Unfortunately, it is not easy to measure g directly. Schwinger was able to check his calculation indirectly as giving part of the contribution to various hyperfine splittings in Hydrogen, such as the Lamb shift. In order to make the comparison, he needed also to be able to get finite predictions out of the divergent integrals, such as the contributions to F_1 in addition to the finite $g-2$ integral. The comparison with data really required a full understanding of all the 1-loop corrections in QED. For this reason, the simplicity of the finite $g-2$ calculation we have just done was not immediately appreciated. Nevertheless, this calculation and the Lamb shift calculation more generally was critically important historically for convincing people that loops in quantum field theory had physical consequences.

The current best measurement is $g = 2.0023193043617 \pm (3 \times 10^{-13})$. The theory calculation has been performed up to 4-loop level. One cannot compare theory to experiment directly, since the theory is expressed as a function of α which cannot be measured more precisely any other way. Therefore $g-2$ is now used to define the renormalized value of the fine structure constant, which comes out to $\alpha = 137.035999070 \pm 9.8 \times 10^{-10}$.